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Fibonacci Sequence and Series by Second Order Difference Operator with Logarithmic Function

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Abstract. In this paper, we introduce second order difference operator with logarithmic function and its inverse by which we obtain Fibonacci sequence and its sum. Some theorems and interesting results on the sum of the terms of advanced Fibonacci sequence are derived. Suitable examples are provided to illustrate our results and verified by using MATLAB.

Key words: Difference operator, Logarithmic function, Fibonacci sequence, Closed form solution.

AMS Subject classification: 39A70, 39A10, 47B39, 65J10, 65Q10.

1. Introduction

In 2011, M.Maria Susai Manuel, et.al, [7] introduced the generalized $\alpha-$difference operator as $\underset{\alpha(\ell)}{\Delta} v(k) = v(k+\ell) - \alpha v(k)$ for the real valued function $v(k)$. In 2014, G.Britto Antony Xavier, et.al, [2] introduced $q-$difference operator as $\Delta_q v(k) = v(qk) - v(k)$, $q \in (0,\infty)$ and obtained finite series solution to the corresponding generalized $q-$difference equation $\Delta_q v(k) = u(k)$. With this backround, in this paper, we obtain advanced Fibonacci sequence and its sum by introducing second order difference operator with logarithmic function.

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2. Advanced Finonacci Sequence and its Sum Obtained by Second Order Difference Equation With Logarithmic Function

Fibonacci and Lucas numbers cover a wide range of interest in modern mathematics as they appear in the comprehensive works of Koshy [6] and Vajda [10]. The $k-$Fibonacci sequence introduced by Falcon and Plaza [3] depends only on one integer parameter k and is defined as

$$F_{k,0} = 0, \quad F_{k,1} = 1 \quad \text{and} \quad F_{k,n+1} = kF_{k,n} + F_{k,n-1}, \quad \text{where} \quad n \geq 1,\ k \geq 1.$$

In particular, if $k = 2$, the Pell sequence is obtained as

$$P_0 = 0, \quad P_1 = 1 \quad \text{and} \quad P_{n+1} = 2P_n + P_{n-1} \quad \text{for} \quad n \geq 1.$$

Here we introduce second order difference operator with logarithmic function $\underset{logk_\alpha}{\Delta} v(k) = v(k) - \alpha_1 log(k_1)v(k-1) - \alpha_2 log(k_2)v(k-2)$ which generates advanced Fibonacci sequence and its sum.

Definition 2.1. A advanced Fibonacci sequence is defined as $F_0 = 1$,

$$F_1 = \alpha_1 logk_1, F_n = \alpha_1 log(k_1 - (n-1))F_{n-1} + \alpha_2 log(k_2 - (n-2))F_{n-2}, n \geq 2 \quad (1)$$

Example 2.2. *Taking $k = 6$, $n = 2$, $\alpha_1 = 2$, $\alpha_2 = 3$, $k_1 = 3$ and $k_2 = 4$ in (1), we get a Fibonacci sequence with logarithmic function $\{1, 2.20, 7.20, 7.24, \cdots\}$.*

Similarly, one can obtain Fibonacci sequences corresponding to each pair $(\alpha_1 logk_1, \alpha_2 logk_2) \in \mathbb{R}^2$.

Definition 2.3. A second order difference operator with logarithmic function on $v(k), k > n+2$, denoted as $\underset{logk}{\Delta} v(k)$, is defined as

$$\underset{logk_\alpha}{\Delta} v(k) = v(k) - \alpha_1 logk_1 v(k-1) - \alpha_2 logk_2 v(k-2), \quad k \in [0, \infty), \quad (2)$$

and its inverse is defined as below;

$$\text{if} \quad \underset{logk_\alpha}{\Delta} v(k) = u(k), \quad \text{then we write} \quad v(k) = \overset{-1}{\underset{logk}{\Delta}} u(k). \quad (3)$$

Lemma 2.4. *Let $v(k), k > n+2$ be a functions of $k \in (-\infty, \infty)$. Then we obtain*

$$\overset{-1}{\underset{logk_\alpha}{\Delta}} a^{sk}\big[1 - \frac{\alpha_1 logk_1}{a^s} - \frac{\alpha_2 logk_2}{a^{2s}}\big] = a^{sk}. \quad (4)$$

Proof. Taking $u(k) = a^{sk}\big[1 - \frac{\alpha_1 logk_1}{a^s} - \frac{\alpha_2 logk_2}{a^{2s}}\big]$ in (2), we obtain $\underset{logk_\alpha}{\Delta} a^{sk} = a^{sk}\big[1 - \frac{\alpha_1 logk_1}{a^s} - \frac{\alpha_2 logk_2}{a^{2s}}\big]$. Now (4) follows from (3). □

Lemma 2.5. *Let $e^{-sk}, k > n+2$ be a function of $k \in (-\infty, \infty)$. Then we have*

$$\underset{logk_\alpha}{\overset{-1}{\Delta}} e^{-sk}\big[1 - \alpha_1 e^s logk_1 - \alpha_2 e^{2s} logk_2\big] = e^{-sk}. \qquad (5)$$

Proof. The proof follows by assuming $a = e^{-1}$ in (4). □

Lemma 2.6. *Let k^k be a function of $k \in (-\infty, \infty)$. Then we have*

$$\underset{logk_\alpha}{\overset{-1}{\Delta}} \big[k^k - \alpha_1 (k-1)^{k-1} logk_1 - \alpha_2 (k-2)^{k-2} logk_2\big] = k^k. \qquad (6)$$

Proof. Taking $v(k) = k^k$ in (2) and using (3). □

Lemma 2.7. *If $v(k) = \underset{logk_\alpha}{\overset{-1}{\Delta}} u(k)$, $F_0 = 1, F_1 = \alpha_1 logk_1$ and $F_{n+1} = F_n \alpha_1 log(k_1 - n) + F_{n-1}\alpha_2 log(k_2 - (n-1))$ for $i = 0, 1, 2, ...$ then*

$$v(k) - F_{n+1} v(k-(n+1)) - F_n \alpha_2 log(k_2 - n) v(k - (n+2)) = \sum_{i=0}^{n} F_i u(k-i). \quad (7)$$

Proof. From (2) and (3), we arrive

$$v(k) = u(k) + \alpha_1 logk_1 v(k-1) + \alpha_2 logk_2 v(k-2). \qquad (8)$$

Replacing k by $k-1$ and then substituting the value of $v(k-1)$ in (8), we get $v(k) = u(k) + F_1 u(k-1) + (F_1\alpha_1 log(k_1-1) + \alpha_2 logk_2) v(k-2) + F_1 \alpha_2 log(k_2-1) v(k-3)$ which gives

$$v(k) = F_0 u(k) + F_1 u(k-1) + F_2 v(k-2) + F_1 \alpha_2 log(k_2 - 1) v(k-3), \qquad (9)$$

where F_0, F_1 and F_2 are given in (1).
Replacing k by $k-2$ in (8) and then substituting $v(k-2)$ in (9), we obtain $v(k) = F_0 u(k) + F_1 u(k-1) + F_2 u(k-2) + F_3 v(k-3) + F_2 \alpha_2 log(k_2 - 2) v(k-4)$, where F_3 is given in (1).
Repeating this process again and again, we get (7). □

Corollary 2.8. *If $v(k), k > n+2$ is a closed form solution of second order difference equation with logarithmic function $\underset{logk_\alpha}{\Delta} v(k) = a^{sk}\big[1 - \frac{\alpha_1 logk_1}{a^s} - \frac{\alpha_2 logk_2}{a^{2s}}\big]$, then we obtain* $a^{sk} - F_{n+1} a^{s(k-(n+1))} - F_n \alpha_2 log(k_2 - n) a^{s(k-(n+2))}$

$$= \sum_{i=0}^{n} F_i a^{s(k-i)} \big[1 - \frac{\alpha_1 log(k_1 - i)}{a^s} - \frac{\alpha_2 log(k_2 - i)}{a^{2s}}\big]. \qquad (10)$$

Proof. The proof follows by applying $v(k) = a^{sk}$ in (7) and using (4). □

The following example is an verification of (10).

Example 2.9. *Taking* $k = 6$, $n = 2$, $s = 2$, $\alpha_1 = 2$, $\alpha_2 = 3$, $a = 2$, $k_1 = 3$ *and* $k_2 = 4$ *in* (10), *we get*

$$2^{12} - F_3 2^6 - 3F_2 log(2)2^4 = \sum_{i=0}^{2} F_i 2^{2(6-i)}\big[1 - \frac{2log(6-i)}{2^2} - \frac{4log(6-i)}{2^4}\big] = 524948.25$$

where $F_0 = 1$, $F_1 = 2.20$, $F_2 = 7.20$ *and* $F_3 = 7.24$.

Corollary 2.10. *Let* $e^{-sk}, k > n+2$ *be a function of* $k \in (-\infty, \infty)$. *Then*

$$e^{-sk} - F_{n+1}e^{-s(k-(n+1))} - \alpha_2 log(k_2 - n)F_n e^{-s(k-(n+2))}$$
$$= \sum_{i=0}^{n} F_i e^{-s(k-i)}\big[1 - \alpha_1 log(k_1 - i)e - \alpha_2 log(k_2 - i)e^{2s}\big]. \quad (11)$$

Proof. Taking $v(k) = e^{-sk}$ and applying (5) in (7), we get (11). □

Example 2.11. *Taking* $k = 5$, $n = 2$, $s = 3$, $\alpha_1 = 3$, $\alpha_2 = 2$, $a = 1.8$, $k_1 = 11$ *and* $k_2 = 21$ *in* (11), *then we obtain*

$$e^{-15} - F_3e^{-6} - 2F_2 log(19)e^{-3} = \sum_{i=0}^{2} F_i e^{-3(5-i)}\big[1 - 3log(11-i)e^3 - 2log(21-i)e^6\big]$$
$$= -66.5$$

where $F_0 = 1$, $F_1 = 9.13$, $F_2 = 86.88$ *and* $F_3 = 809.51$.

Lemma 2.12. *Let* $t \in \mathbb{N}(0), k > n+2$. *Then a closed form solution of the second order difference equation logarithmic function* $v(k) - \sum_{p=1}^{2} \alpha_p log k_p v(k-p) = \big[k^t - \sum_{p=1}^{2} \alpha_p log k_p (k-p)^t\big]$ *is*

$$\mathop{\Delta}_{logk_\alpha}^{-1} \big[k^t - \alpha_1 log k_1 (k-1)^t - \alpha_2 log k_2 (k-2)^t\big] = k^t \quad (12)$$

Proof. Taking $v(k) = k^t$ in (2) and using (3), we get (12). □

Corollary 2.13. *If* $v(k) = \mathop{\Delta}_{logk_\alpha}^{-1} \big[k^t - \sum_{p=1}^{2} \alpha_p log k_p (k-p)^t\big], k > n+2$ *is the closed form solution given in (12), then* $k^t - F_{n+1}(k-n-1)^t - F_n \alpha_2 log(k_2 - n)(k-n-2)^t$

$$= \sum_{i=0}^{n} F_i\big[(k-i)^t - \alpha_1 log(k_1 - i)[k-(i+1)]^t - \alpha_2 log(k_2 - i)[k-(i+2)]^t\big]. \quad (13)$$

Proof. The proof follows by taking $u(k) = k^t - \sum_{p=1}^{2} \alpha_p log k_p (k-p)^t$ in (7). □

Example 2.14. *Let* $k = 1011$, $n = 3$, $t = 4$, $k_1 = 21$, $k_2 = 9$, $\alpha_1 = 5$, $\alpha_2 = 3$ *in Corollary* (2.13). *Then*

$$\sum_{i=0}^{3} F_i u(1011-i) = v(1011) - F_4 v(1007) - 3log(6)F_3 v(1006) = -10438211162429716.$$

where $u(k) = k^t - \alpha_1 log k_1 (k-1)^t - \alpha_2 log k_2 (k-2)^t$, $F_0 = 1$, $F_1 = 9.13$, $F_2 = 86.48$, $F_3 = 801.89$ *and* $F_4 = 7289.82$.

Corollary 2.15. *Let $v(k)$, $k > n+2$ be a solution of second order difference equation with logarithmic functions*
$v(k) - \sum_{p=1}^{2} \alpha_p log k_p v(k-p) = [\frac{k^t}{\sqrt{log(bk)}} - \sum_{p=1}^{2} \alpha_p log(k_p - p) \frac{(k-p)^t}{\sqrt{log(b(k-p))}}].$
Then we have

$$\frac{k^t}{\sqrt{log(bk)}} - F_{n+1} \frac{(k-n-1)^t}{\sqrt{log(b(k-n-1))}} - \alpha_2 log(k_2 - n) \frac{(k-n-2)^t}{\sqrt{log(b(k-n-2))}}$$
$$= \sum_{i=0}^{n} F_i \Big[\frac{(k-i)^t}{\sqrt{log(b(k-i))}} - \sum_{p=1}^{2} \alpha_p log(k_p - p) \frac{(k-i-p)^t}{\sqrt{log(b(k-i-p))}} \Big]. \quad (14)$$

Proof. The proof follows by taking $v(k) = \frac{k^t}{\sqrt{log(bk)}}$ in the Theorem 2.7. □

Example 2.16. *Let $k = 14$, $n = 3$, $b = 10$, $t = 7$, $\alpha_1 = 15$, $\alpha_2 = 19$, $k_1 = 29$, $k_2 = 32$ in Corollary 2.28. Then we obtain*
$\frac{21^4}{\sqrt{log(231)}} - F_4 \frac{(17)^4}{\sqrt{log(187)}} - 10.5 log(k_2 - 3) \frac{(16)^4}{\sqrt{log(176)}}$
$= \sum_{i=0}^{3} F_i \Big[\frac{(21-i)^4}{\sqrt{log(11(21-i))}} - \sum_{p=1}^{2} \alpha_p log(k_p - p) \frac{(21-i-p)^4}{\sqrt{log(11(21-i-p))}} \Big]$
$= -226630264497941.12,$

where $F_0 = 1$, $F_1 = 50.51$, $F_2 = 2590.47$, $F_3 = 131361.81$ and $F_4 = 6587244.76$.

Lemma 2.17. *If $v(k), k > n+2$ is a closed form solution of the second order difference equation with logarithmic functions*
$v(k) - \sum_{p=1}^{2} \alpha_p log k_p v(k-p) = k^t sin(sk) - \sum_{p=1}^{2} \alpha_p log(k_p - p)(k-p)^t sin(s(k-p)),$
then

$$k^t sin(sk) - F_{n+1}(k-n-1)^t sin(s(k-n-1)) - \alpha_2 log(k_2 - n)(k-n-2)^t sin(s(k-n-2)) =$$
$$\sum_{i=0}^{n} F_i \Big[(k-i)^t sin(s(k-i)) - \sum_{p=1}^{2} \alpha_p log(k_p - p)(k-i-p)^t sin(s(k-i-p)) \Big]. \quad (15)$$

Proof. Taking $u(k) = \big[k^t sin(sk) - \sum_{p=1}^{2} \alpha_p log(k_p - 1)(k-p)^t sin(s(k-p))\big]$ in Theorem 2.7 and using (4), we get 15. □

Corollary 2.18. *A closed form solution of the logarithmic second order difference equation logarithmic function* $\underset{logk_\alpha}{\Delta} v(k) = k^4 sin(sk) - \sum_{p=1}^{2} \alpha_p logk_p (k-p)^4 sin(s(k-p))$ *and hence we have*

$$k^4 sinsk - F_{n+1}(k-(n+1))^4 sin(s(k-(n+1))) - F_n \alpha_2 log(k_2 - n)(k-(n+2))^4 \times \sin(s(k-(n+2))) = \sum_{i=0}^{n} F_i \big[(k-i)^4 - \sum_{p=1}^{2} \alpha_p \log(k_p - p)(k-(i+p))^4 \sin s(k-p-i)\big]. \quad (16)$$

Proof. The proof follows by taking $t = 4$ in Theorem 2.17. □

Example 2.19. *Let* $k = 9$, $a = 3$, $n = 3$, $\alpha_1 = 2$, $s = 2$, $\alpha_2 = 3$, $k_1 = 29.55$, $k_2 = 15.9$ *in Corollary* (2.18). *We have*

$$9^4 sin18 - F_4(5)^4 sin(10) - 3log(12.9)(4)^4 sin(8)$$
$$= \sum_{i=0}^{3} F_i \big[(9-i)^4 - \sum_{p=1}^{2} \alpha_p log(k_p - p)(9-(i+p))^4 sins(9-p-i)\big]$$
$$= 2421452716882.48,$$

where $F_0 = 1$, $F_1 = 6.77$, $F_2 = 53.69$, $F_3 = 410.99$ *and* $F_4 = 3119.24$.

Corollary 2.20. *A closed form solution of the second order difference equation with logarithmic functions*

$$v(k) - \sum_{p=1}^{2} \alpha_p log(k_p - p) v(k-p) = k^t tan(bk) - \sum_{p=1}^{2} \alpha_p log(k_p - p)(k-p)^t tanb(k-p)$$

is given by

$$k^t tan(bk) - F_{n+1}(k-(n+1))^t tan(b(k-(n+1))) - \alpha_2 F_n log(k_2 - n)(k-(n+2))^t tan(b(k-(n+2)))$$
$$= \sum_{i=0}^{n} F_i \big[(k-i)^t tan(b(k-i)) - \sum_{p=1}^{2} \alpha_p log(k_p - p)(k-i-p)^t tan(b(k-i-p))\big]. \quad (17)$$

Proof. Taking $v(k) = k^t tan(bk)$ in Theorem 2.7, we get (17). □

Example 2.21. *Let* $k = 2.0091$, $n = 3$, $t = 5$, $a = 4$, $b = 5$, $\alpha_1 = 3$, $\alpha_2 = 2$, $k_1 = 34.85$ *and* $k_2 = 12.5$ *in Corollary* (2.18). *We have*

$$(2.0091)^5 tan(10.0455) - F_4(-1.9909)^5 tan(-9.955) - 2F_3 log(9.5)(-2.9909)^5 tan(-14.9545)$$
$$= \sum_{i=0}^{3} F_i [(2.0091-i)^5 tan(5(2.0091-i)) - \sum_{p=1}^{2} \alpha_p log(k_p - p)(2.0091-i-p)^5 \times \times tan(5(2.0091-i-p))] = 1043823.96,$$

where $F_0 = 1$, $F_1 = 10.65$, $F_2 = 117.61$, $F_3 = 1284.11$ *and* $F_4 = 13886.16$.

Corollary 2.22. *A closed form solution of the second order difference equation with logarithmic functions*
$v(k)-\sum\limits_{p=1}^{2}\alpha_p log(k_p-p)v(k-p) = e^{-sk}log(bk)-\sum\limits_{p=1}^{2}\alpha_p log(k_p-p)e^{s(k-p)}log(b(k-p))$
is given by

$$e^{sk}log(bk) - F_{n+1}e^{s(k-n-1)}log(b(k-n-1)) - \alpha_2 F_n log(k_2-n)e^{s(k-n-2)}log(b(k-n-2))$$
$$= \sum_{i=0}^{n} F_i e^{s(k-i)}\big[log(b(k-i)) - \sum_{p=1}^{2}\alpha_p log(k_p-p)e^{-sp}log(b(k-i-p))\big]. \quad (18)$$

Proof. Taking $v(k) = e^{sk}log(bk)$ in Theorem 2.7, we get (18). □

Example 2.23. *Let* $k = 7$, $n = 3$, $s = 1.5$, $b = 2$, $\alpha_1 = 15$, $\alpha_2 = 10$, $k_1 = 18$ *and* $k_2 = 15$ *in Corollary* (2.18). *We have*
$e^{10.5}log(14) - F_4e^{4.5}log(6) - 10F_3log(12)e^3log(4)$
$= \sum\limits_{i=0}^{3} F_i e^{1.5(7-i)}\big[log(2(7-i)) - \sum\limits_{p=1}^{2}\alpha_p log(k_p-p)e^{-1.5p}log(2(7-i-p))\big]$
$= -579152724.29$, *where* $F_0 = 1$, $F_1 = 43.36$, $F_2 = 1869.61$, $F_3 = 78899.26$ *and* $F_4 = 3252901.95$.

Lemma 2.24. *Let* $v(k), k > n+2$ *be a solution of the second order difference equation with logarithmic functions. Then we have,*
$v(k) - \sum\limits_{i=1}^{2}\alpha_i log(k_i-i)v(k-i) = k^{(t)}cosech(k) - \sum\limits_{i=1}^{2}\alpha_i logk_i(k-i)^{(t)}cosech(k-i)$,
then we have, $k^{(t)}cosech(k) - F_{n+1}(k-n-1)^t cosech(k-n-1) - \alpha_2 log(k_2-n)(k-n-2)^t cosech(k-n-2) = \sum\limits_{i=0}^{n} F_i \times$

$$\big[(k-i)^{(t)}cosech(k-i) - \sum_{p=1}^{2}\alpha_p log(k_p-p)(k-i-p)^{(t)}cosech(k-i-p)\big]. \quad (19)$$

Proof. Taking $v(k) = k^{(t)}cosech(k)$ in Theorem 2.7 and using (4), we get 19. □

Example 2.25. *Let* $k = 1.5$, $b = 5$, $n = 3$, $t = 4$, $\alpha_1 = 12$, $\alpha_2 = 20$, $k_1 = 50$, $k_2 = 34$ *in Theorem 2.24. We have,*
$(1.5)^{(4)}cosech(1.5) - F_4(-2.5)^{(4)}cosech(-2.5) - 20log(31)(-3.5)^{(4)}cosech(3.5)$
$= \sum\limits_{i=0}^{3} F_i\big[(1.5-i)^{(4)}cosech(1.5-i) - \sum\limits_{p=1}^{2}\alpha_p log(k_p-p)(1.5-i-p)^{(4)}cosech(1.5-i-p)\big] = 8547.97$,
where $F_0 = 1$, $F_1 = 46.94$, $F_2 = 2262.91$, $F_3 = 108405.04$ *and* $F_4 = 5165357.99$.

Corollary 2.26. *Let $v(k)$, $k > n+2$ be a solution of second order difference equation with logarithmic functions*
$v(k) - \sum_{p=1}^{2} \alpha_p log k_p v(k-p) = \big[log(log(bk))k^{(t)} - \sum_{p=1}^{2} \alpha_p log(k_p - p) log(log(b(k-p)))(k-p)^{(t)}\big]$.
Then we have

$$log(log(bk))k^{(t)} - F_{n+1} log(log(b(k-n-1)))(k-n-1)^{(t)} - \alpha_2 log(k_2 - n) log(log(b(k-n-2)))(k-n-2)^{(t)}$$

$$= \sum_{i=0}^{n} F_i \big[log(log(b(k-i)))(k-i)^{(t)} - \sum_{p=1}^{2} \alpha_p log(k_p - p) log(log(b(k-i-p)))(k-i-p)^{(t)}\big]. \quad (20)$$

Proof. Taking $u(k) = log(log(bk))k^{(t)} - \sum_{p=1}^{2} \alpha_p log(k_p - p) log(log(b(k-p)))(k-p)^{(t)}$ in Theorem 2.7 and using (4), we get (20). □

Example 2.27. *Let $k = 16$, $n = 3$, $b = 10$, $t = 4$, $\alpha_1 = 15$, $\alpha_2 = 19$, $k_1 = 29$, $k_2 = 32$ in Corollary 2.26. Then we obtain*
$log(log(160))16^{(4)} - F_4 log(log(120))(12)^{(4)} - 19 log(k_2 - 3) log(log(110))(11)^{(4)}$
$= \sum_{i=0}^{3} F_i \big[log(log(10(16-i)))(16-i)^{(4)} - \sum_{p=1}^{2} \alpha_p log(k_p - p) log(log(10(16-i-p)))(16-i-p)^{(4)}\big]$
$= -225566400698.78$, *where* $F_0 = 1$, $F_1 = 50.51$, $F_2 = 2590.47$, $F_3 = 131361.81$ *and* $F_4 = 6587244.76$.

Corollary 2.28. *Let $v(k)$, $k > n+2$ be a solution of second order difference equation with logarithmic functions*
$v(k) - \sum_{p=1}^{2} \alpha_p log k_p v(k-p) = \big[\frac{k^{(t)}}{log(log(bk))} - \sum_{p=1}^{2} \alpha_p log(k_p - p) \frac{(k-p)^{(t)}}{log(log(b(k-p)))}\big]$.
Then we have

$$\frac{k^{(t)}}{log(log(bk))} - F_{n+1} \frac{(k-n-1)^{(t)}}{log(log(b(k-n-1)))} - \alpha_2 log(k_2 - n) \frac{(k-n-2)^{(t)}}{log(log(b(k-n-2)))}$$

$$= \sum_{i=0}^{n} F_i \big[\frac{(k-i)^{(t)}}{log(log(b(k-i)))} - \sum_{p=1}^{2} \alpha_p log(k_p - p) \frac{(k-i-p)^{(t)}}{log(log(b(k-i-p)))}\big]. \quad (21)$$

Proof. The proof follows by taking $v(k) = \frac{k^{(t)}}{log(log(bk))}$ in the Theorem 2.7. □

Example 2.29. *Let $k = 21$, $n = 3$, $b = 11$, $t = 4$, $\alpha_1 = 14$, $\alpha_2 = 10.5$, $k_1 = 27$, $k_2 = 29$ in Corollary 2.28. Then we obtain*
$\frac{21^{(4)}}{log(log(231))} - F_4 \frac{(17)^{(4)}}{log(log(187))} - 10.5 log(k_2 - 3) \frac{(16)^{(4)}}{log(log(176))}$
$= \sum_{i=0}^{3} F_i \big[\frac{(21-i)^{(4)}}{log(log(11(21-i)))} - \sum_{p=1}^{2} \alpha_p log(k_p - p) \frac{(21-i-p)^{(4)}}{log(log(11(21-i-p)))}\big]$

$= -242342850538.15,$

where $F_0 = 1,\ F_1 = 46.14,\ F_2 = 2140.03,\ F_3 = 98053.51$ *and* $F_4 = 4436729.18.$

Conclusion: We obtained summation formula to advanced Fibonacci sequence by introducing second order difference operator with logarithmic function and have derived certain results on closed and summation form solution of second order difference equation which will be used to our further research.

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