Original Research Article

J. Comp. Matha. 01 (2017) 2456–8686

SACRED HEART RESEARCH PUBLICATIONS

Journal of Computational Mathematica

Journal homepage: www.shcpub.edu.in

ISSN: 2456-8686

Equitable Coloring of Some Cycle Related Graphs

[1]S. K. Vaidya, and [2]Rakhimol V. Isaac

Received on 15 Jan 2017, Accepted on 08 April 2017

Abstract. The graph is called *equitably k-colorable* if the vertex set of the graph can be partitioned into k non empty independent sets $V_1, V_2, \ldots, V_k$ such that $|\,|V_i| - |V_j|\,| \leq 1$, for every i and j. The smallest integer k for which the graph G is equitably k- colorable is called the equitable chromatic number of graph G and is denoted by $\chi_e(G)$. If the connected graph G is neither a complete graph nor an odd cycle then the Equitable Coloring Conjecture (ECC) states that $\chi_e(G) \leq \triangle(G)$. In this work we investigate the equitable chromatic number of some cycle related graphs like middle graph, shadow graph and splitting graph of cycle.

Keywords: Equitable coloring, Equitable chromatic number, Middle graph, Shadow graph, Splitting graph.

AMS Subject classification: 05C15, 05C76

1. Introduction

We begin with finite, connected, simple and undirected graph with vertex set $V(G)$ and edge set $E(G)$. For any undefined term in graph theory we refer to Bondy and Murty [1]. A *proper k-coloring* of a graph G is a function $c : V(G) \to \{1, 2, ..., k\}$ such that $c(u) \neq c(v)$ for all $uv \in E(G)$. The chromatic number $\chi(G)$ is the minimum number k for which G admits proper k-coloring. There are many variants of proper coloring like b-coloring, total coloring, dominator coloring, equitable coloring etc. The present work is intended to report some investigations on equitable coloring of graph.

The graph G is called *equitably k-colorable* if the vertex set of G can be partitioned into k non empty independent sets $V_1, V_2, \ldots, V_k$ such that $|\,|V_i| - |V_j|\,| \leq 1$, for every

[1]*Corresponding Author: E-mail: samirkvaidya@yahoo.co.in, rakhiisaac@yahoo.co.in

[1,2]Department of Mathematics, Saurashtra University, Rajkot - 360005, Gujarat, India

[2]Department of Mathematics, Christ College, Rajkot - 360005, Gujarat, India.

i and j. The smallest integer k for which G is equitably k- colorable is called the equitable chromatic number of G and is denoted by $\chi_e(G)$. If the connected graph G is neither a complete graph nor an odd cycle then the Equitable Coloring Conjecture (ECC) states that $\chi_e(G) \leq \triangle(G)$ [8, 6].

The notion of equitable coloring is first introduced by Meyer [8]. The equitable coloring of trees and bipartite graphs are studied by Chen et al [2] and Lih et al [7] respectively. The equitable coloring of graph products is discussed in [4].

Proposition 1.1. [5] *If G and G' are simple graphs on the same set of vertices and $E(G) \subseteq E(G')$, then $\chi_e(G) \leq \chi_e(G')$.*

Proposition 1.2. [3] *If G contains a clique of order n then $\chi(G) \geq n$.*

Proposition 1.3. [8] *Since an equitable coloring is a proper coloring,*
$\chi_e(G) \geq \chi(G)$

2. Main Results

Definition 2.1. The middle graph $M(G)$ of a graph G is the graph whose vertex set $V(G) \cup E(G)$ in which two vertices are adjacent if and only if either they are adjacent edges of G or one is a vertex of G and the other is an edge incident on it.

Theorem 2.2. $\chi_e(M(C_n)) = 3$, *for all n.*

Proof. Let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(C_n) = \{e_1, e_2, \ldots, e_n\}$ where $e_i = v_i v_{i+1}$; $1 \leq i \leq n-1$ and $e_n = v_n v_1$. By the definition of middle graph, $V(M(C_n)) = V(C_n) \cup E(C_n)$ and $E(M(C_n)) = \{v_i e_i; 1 \leq i \leq n\} \cup \{e_i v_{i+1}; 1 \leq i \leq n-1\} \cup \{e_n v_1\} \cup \{e_i e_{i+1}; 1 \leq i \leq n-1\} \cup \{e_n e_1\}$. Thus $|V(M(C_n))| = 2n$ and $|E(M(C_n))| = 3n$.

As $M(C_n)$ contains a clique of order 3, $\chi(M(C_n)) \geq 3$ according to Proposition 1.2 and so,

$$\chi_e(M(C_n)) \geq 3. \tag{1}$$

<u>**Case 1**</u>: *when $n \equiv 0(mod\ 3)$:*

Consider the color function $c : V(M(C_n)) \to \mathbb{N}$ as

$c(v_{3k}) = c(e_{3k-2}) = 1$; $c(v_{3k-1}) = c(e_{3k}) = 3$; $c(v_{3k-2}) = c(e_{3k-1}) = 2$ where

$k = 1, 2, \ldots, \frac{n}{3}$.

Now, partition the vertex set $V(M(C_n))$ as

$V_1 = \{e_1, e_4, \ldots, e_{n-2}, v_3, v_6, v_9, \ldots, v_n\}$,

$V_2 = \{e_2, e_5, e_8, \ldots, e_{n-1}, v_1, v_4, v_7, \ldots, v_{n-2}\}$,

$V_3 = \{e_3, e_6, e_9, \ldots, e_n, v_2, v_5, v_8, \ldots, v_{n-1}\}$.

Clearly, V_1, V_2 and V_3 are independent sets of $M(C_n)$. Also, $|V_1|=|V_2|=|V_3|=\frac{2n}{3}$. Thus, $\chi_e(M(C_n)) \leq 3$.

<u>Case 2</u>: *when* $n \equiv 1(mod\ 3)$:

Consider the color function $c : V(M(C_n)) \to \mathbb{N}$ as

$c(v_n) = 1$, $c(e_n) = 2$, $c(v_1) = 3$, $c(v_{3k+1}) = 2$; $k = 1, 2, \ldots, \frac{n-4}{3}$.

$c(v_{3k}) = c(e_{3k-2}) = 1$; $c(e_{3k-1}) = 2$; $c(e_{3k}) = c(v_{3k-1}) = 3$; $k = 1, 2, \ldots, \frac{n-1}{3}$.

Now, partition the vertex set $V(M(C_n))$ as

$V_1 = \{e_1, e_4, e_7, \ldots, e_{n-3}, v_3, v_6, v_9, \ldots, v_{n-1}, v_n\}$,

$V_2 = \{e_2, e_5, e_8, \ldots, e_{n-2}, e_n, v_4, v_7, v_{10}, \ldots, v_{n-3}\}$,

$V_3 = \{e_3, e_6, e_9, \ldots, e_{n-1}, v_1, v_2, v_5, v_8, \ldots, v_{n-2}\}$.

Now, $|V_1| = \frac{n-1}{3} + \frac{n-1}{3} + 1 = \frac{2n+1}{3}$, $|V_2| = \frac{n-1}{3} + 1 + \frac{n-4}{3} = \frac{2n-2}{3}$, and $|V_3| = \frac{n-1}{3} + 1 + \frac{n-1}{3} = \frac{2n+1}{3}$.

Clearly, V_1, V_2 and V_3 are independent sets of $M(C_n)$.

Also, $|V_1|=|V_3|$ and $\big|\,|V_1| - |V_2|\,\big| = \big|\,|V_2| - |V_3|\,\big| = \big|\frac{2n+1}{3} - (\frac{2n-2}{3})\big| = 1$. It holds the inequality $|\,|V_i| - |V_j|\,| \leq 1$, for every i and j. Thus, $\chi_e(M(C_n)) \leq 3$.

<u>Case 3</u>: *when* $n \equiv 2(mod\ 3)$:

Consider the color function $c : V(M(C_n)) \to \mathbb{N}$ as

$c(v_1) = 3$; $c(e_{3k-2}) = 1$; $c(e_{3k-1}) = 2$; $c(v_{3k-1}) = 3$ where $k = 1, 2, \ldots, \frac{n+1}{3}$.

$c(v_{3k}) = 1$; $c(v_{3k+1}) = 2$; $c(e_{3k}) = 3$ where $k = 1, 2, \ldots, \frac{n-2}{3}$.

Now, partition the vertex set $V(M(C_n))$ as

$V_1 = \{e_1, e_4, e_7, \ldots, e_{n-1}, v_3, v_6, v_9, \ldots, v_{n-2}\}$,

$V_2 = \{e_2, e_5, e_8, \ldots, e_n, v_4, v_7, v_{10}, \ldots, v_{n-1}\}$,

$V_3 = \{e_3, e_6, e_9, \ldots, e_{n-2}, v_1, v_2, v_5, v_8, \ldots, v_n\}$.

Now, $|V_1| = \frac{n+1}{3} + \frac{n-2}{3} + 1 = \frac{2n-1}{3}$, $|V_2| = \frac{n+1}{3} + \frac{n-2}{3} = \frac{2n-1}{3}$, and $|V_3| = \frac{n-2}{3} + 1 + \frac{n+1}{3} = \frac{2n+2}{3}$.

Clearly, V_1, V_2 and V_3 are independent sets of $M(C_n)$.
Also, $|V_1|$=$|V_2|$ and $|\,|V_1| - |V_3|\,|$= $|\,|V_2| - |V_3|\,| = |\frac{2n-1}{3} - (\frac{2n+2}{3})| = 1$. It holds the inequality $|\,|V_i| - |V_j|\,| \leq 1$, for every i and j. Thus, $\chi_e(M(C_n)) \leq 3$.

Thus,

$$\chi_e(M(C_n)) \leq 3, for\ all\ n. \tag{2}$$

Therefore from (1) and (2) we get, $\chi_e(M(C_n)) = 3$. □

Definition 2.3. The shadow graph $D_2(G)$ of a connected graph G is constructed by taking two copies of G, say G' and G''. Join each vertex u' in G' to the neighbors of the corresponding vertex u'' in G''.

Theorem 2.4.

$$\chi_e(D_2(C_n)) = \begin{cases} 2, & n\ is\ even \\ 3, & n\ is\ odd \end{cases}$$

Proof. Let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(C_n) = \{e_1, e_2, \ldots, e_n\}$ where $e_i = v_i v_{i+1}$; $1 \leq i \leq n-1$ and $e_n = v_n v_1$. The shadow graph of C_n, $D_2(C_n)$, has two copies, say C'_n and C''_n, with $V(D_2(C_n)) = \{v'_i, v''_i; 1 \leq i \leq n\}$. In $D_2(C_n)$, $N(v'_i) = N(v''_i)$ and v'_i and v''_i are non adjacent vertices.

<u>**Case 1**</u>: *when n is even*:

Consider the color function $c : V(D_2(C_n)) \to \mathbb{N}$ as
$c(v'_{2k-1}) = c(v''_{2k-1}) = 1$ and $c(v'_{2k}) = c(v''_{2k}) = 2$.
Now, partition the vertex set of $D_2(C_n)$ as
$V_1 = \{v'_1, v''_1, v'_3, v''_3, \ldots, v'_{n-1}, v''_{n-1}\}$ and
$V_2 = \{v'_2, v''_2, v'_4, v''_4, \ldots, v'_n, v''_n\}$.
Clearly, V_1 and V_2 are independent sets of $D_2(C_n)$ and $|\,|V_1| - |V_2|\,| = 1$. It holds the inequality $|\,|V_i| - |V_j|\,| \leq 1$ for every pair (i, j). Thus $\chi_e(D_2(C_n)) = 2$.

<u>**Case 2**</u>: *when n is odd*:

As $D_2(C_n)$ contains an odd cycle, it is not a bipartite graph. So it is not possible to partition the vertex set of $D_2(C_n)$ into two independent sets. Therefore $\chi_e(D_2(C_n)) \neq 2$. Thus we have the following three subcases:

Subcase 1: $n \equiv 0(mod\ 3)$:

Consider the color function $c : V(D_2(C_n)) \to \mathbb{N}$ as

$c(v'_{3k-2}) = c(v''_{3k-2}) = 1$;

$c(v'_{3k-1}) = c(v''_{3k-1}) = 2$;

$c(v'_{3k}) = c(v''_{3k}) = 3$; $k = 1, 2, \ldots, \frac{n}{3}$.

Now, partition the vertex set of $D_2(C_n)$ as

$V_1 = \{v'_{3k-2}, v''_{3k-2}\}$, $V_2 = \{v'_{3k-1}, v''_{3k-1}\}$ and $V_3 = \{v'_{3k}, v''_{3k}\}$; $k = 1, 2, \ldots, \frac{n}{3}$.

Clearly, V_1, V_2 and V_3 are independent sets of $D_2(C_n)$. Also, $|V_1|=|V_2|=|V_3|=\frac{2n}{3}$. Thus, $\chi_e(D_2(C_n)) = 3$.

Subcase 2: $n \equiv 1(mod\ 3)$:

Consider the color function $c : V(D_2(C_n)) \to \mathbb{N}$ as

$c(v'_1) = c(v'_{n-2}) = 1$; $c(v'_{3k}) = c(v''_{3k}) = 1$; $k = 1, 2, \ldots, \frac{n-4}{3}$.

$c(v'_n) = 2$, $c(v'_{3k-1}) = 2$; $k = 1, 2, \ldots, \frac{n-4}{3}$.

$c(v''_n) = 2$, $c(v''_{3k-1}) = 2$; $k = 1, 2, \ldots, \frac{n-1}{3}$.

$c(v'_{n-1}) = 3$, $c(v'_{3k+1}) = 3$; $k = 1, 2, \ldots, \frac{n-4}{3}$.

$c(v''_{n-1}) = 3$, $c(v''_{3k-2}) = 3$; $k = 1, 2, \ldots, \frac{n-1}{3}$.

Now, partition the vertex set of $D_2(C_n)$ as

$V_1 = \{v'_1, v'_{n-2}, v'_{3k}, v''_{3k}; k = 1, 2, \ldots, \frac{n-4}{3}\}$,

$V_2 = \{v'_n, v''_n, v'_{3k-1}(k = 1, 2, \ldots, \frac{n-4}{3}), v''_{3k-1}(k = 1, 2, \ldots, \frac{n-1}{3})\}$, and

$V_3 = \{v'_{n-1}, v''_{n-1}, v'_{3k+1}(k = 1, 2, \ldots, \frac{n-4}{3}), v''_{3k-2}(k = 1, 2, \ldots, \frac{n-1}{3})\}$.

Clearly, V_1, V_2 and V_3 are independent sets of $D_2(C_n)$.

Also, $|V_1| = \frac{2n-2}{3}$, $|V_2| = \frac{2n+1}{3}$ and $|V_3| = \frac{2n+1}{3}$. Now, $|V_2| = |V_3|$ and $\big|\, |V_1| - |V_2| \,\big| = \big|\, |V_1| - |V_3| \,\big| = |\frac{2n-2}{3} - (\frac{2n+1}{3})| = 1$. It holds the inequality $\big|\, |V_i| - |V_j| \,\big| \leq 1$, for every pair (i, j). Thus, $\chi_e(D_2(C_n)) = 3$.

Subcase 3: $n \equiv 2(mod\ 3)$:

Consider the color function $c : V(D_2(C_n)) \to \mathbb{N}$ as

$c(v'_{3k-2}) = 1$; $k = 1, 2, \ldots, \frac{n+1}{3}$.

$c(v''_{3k+1}) = 1$; $k = 1, 2, \ldots, \frac{n-2}{3}$

$c(v'_{3k-1}) = c(v''_{3k-1}) = 2$; $k = 1, 2, \ldots, \frac{n+1}{3}$.
$c(v''_1) = 3$; $c(v'_{3k}) = c(v''_{3k}) = 3$; $k = 1, 2, \ldots, \frac{n-2}{3}$.
Now, partition the vertex set of $D_2(C_n)$ as

$V_1 = \{v'_1, v'_4, v'_7, \ldots, v'_{n-1}, v''_4, v''_7, \ldots, v''_{n-1}\}$,

$V_2 = \{v'_2, v'_5, v'_8, \ldots, v'_n, v''_2, v''_5, \ldots, v''_n\}$,

$V_3 = \{v'_3, v'_6, \ldots, v'_{n-2}, v''_1, v''_3, v''_6, \ldots, v''_{n-2}\}$. Now, V_1, V_2 and V_3 are independent sets of $D_2(C_n)$.

Also, $|V_1| = \frac{n+1}{3} + \frac{n-2}{3} = \frac{2n-1}{3}$, $|V_2| = \frac{2(n+1)}{3}$ and $|V_3| = \frac{2(n-2)}{3} + 1 = \frac{2n-1}{3}$.
Now, $|V_1| = |V_3|$ and $\left| |V_2| - |V_3| \right| = \left| |V_2| - |V_1| \right| = \left| \frac{2n+2}{3} - (\frac{2n-1}{3}) \right| = 1$. It holds the inequality $\left| |V_i| - |V_j| \right| \leq 1$, for every pair (i, j). Thus, $\chi_e(D_2(C_n)) = 3$. □

Definition 2.5. The splitting graph $S'(G)$ of a connected graph G is obtained by adding new vertex v' corresponding to each vertex v of G such that $N(v) = N(v')$ where $N(v)$ and $N(v')$ are the neighborhood sets of v and v' respectively.

The following theorem can be proved by the arguments analogous to the previous Theorem 2.4.

Theorem 2.6.

$$\chi_e(S'(C_n)) = \begin{cases} 2, & n \text{ is even} \\ 3, & n \text{ is odd} \end{cases}$$

Proof. We have $V(S'(C_n)) = V(D_2(C_n))$ and $E(S'(C_n)) \subset E(D_2(C_n))$. Then by Proposition 1.1, $\chi_e(S'(C_n)) \leq \chi_e(D_2(C_n))$. Now, we can assign the same coloring as in $D_2(C_n)$ and hence, $\chi_e(S'(C_n)) = \chi_e(D_2(C_n))$. □

3. Conclusion

The equitable chromatic number of some standard graphs like cycle, path, wheel etc are known. We have investigated the equitable chromatic number for the larger graphs such as $M(C_n), D_2(C_n)$ and $S'(C_n)$ obtained from cycle C_n.

References

[1] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, London: Macmillan, 1976.

[2] B. l. Chen and K. W. Lih, "Equitable Coloring of Trees", Journal of Combinatorial Theory, Series B, 61(1994), 83-87.

[3] J. Clark and D.A. Holton, *A First Look at Graph Theory*, World Scientific, 1995.

[4] H. Furmańczyk, "Equitable Coloring of Graph Products", Opuscula Mathematica, 26(1) (2006), 31-44.

[5] K. Kaliraj, "On Equitable coloring of Graphs", Ph. D. Thesis, Anna University, Chennai, (2013).

[6] T. Karthick, "Note on Equitable coloring of Graphs", Australian Journal of Combinatorics, 59(2) (2014), 251-259.

[7] K. W. Lih and P. L. Wu, "On Equitable coloring of Bipartite Graphs", Discrete Mathematics, 151 (1996), 155-160.

[8] W. Meyer, "Equitable Coloring", The American Mathematical Monthly, 80(8) (1973), 920-922.