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Fibonacci Sequence Generated from Three-Dimensional q-Difference Equation

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Abstract. In this research work, the authors define three dimensional q-difference operator and present some significant results using the inverse of the three dimensional q-difference operator. Also the summation solution and the closed form solutions of polynomials and logarithmic functions using three dimensional q-difference equation have been obtained. This work also includes generalized product formula for polynomial, reciprocal of polynomial and logarithmic functions using inverse of three dimensional q-difference operator. Consequently, relevant examples are being given to investigate the results.

Key words: Fibonacci numbers, higher order q-difference operator and Summation solution.

AMS Subject classification: 39A10, 47B39, 39A70, 49M.

1. Introduction

The study of q-difference equations, initiated at the beginning of the twentieth century in intensive works especially by Jackson [6], Carmichael [5] and other authors such as Poincare, Picard, Ramanujan, is a very interesting field in difference equations. In the last few decades, it has evolved into a multidisciplinary subject and plays an important role in several fields of physics. In 1984, Jerzy Popenda [3] introduced a particular type of difference operator Δ_α defined on $u(k)$ as $\Delta_\alpha u(k) = u(k+1) - \alpha u(k)$. Recently, G.Britto Antony Xavier

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et al. [2] have got the solution of the generalized q-difference equation $\Delta_q^t v(k) = u(k),\ k \in (-\infty, \infty)$ and $q \neq 1$. In [4], the authors introduced q-alpha difference operator defined as

$$\underset{(q)\alpha}{\Delta} u(k) = u(qk) - \alpha u(k), \tag{1}$$

and then extended to generalized higher oredr q-alpha difference equation

$$\underset{(q_1)\alpha_1}{\Delta}\Big(\underset{(q_2)\alpha_2}{\Delta}\Big(\cdots \underset{(q_t)\alpha_t}{\Delta}\big(v(k)\big)\cdots\Big)\Big) = u(k), \quad k \in (-\infty, \infty), \tag{2}$$

and obtained finite q-alpha multi-series formula and finite higher order q-alpha series formula.

With this background in this article, we introduce three dimensional q-difference operator and derive formula for finite series to polynomials and logarithmic functions by finding the numerical and closed form solutions of the three dimensional q-difference equation.

2. Three dimensional q-difference operator

Before stating and proving our results, we present basic definitions and preliminary results which will be used for the subsequent discussions.

Definition 2.1. *Let p_1 and p_2 be the fixed reals and $(k_1, k_2, k_3) \in \Re^3$. Then the three-dimensional q-difference operator $\underset{(p_1,p_2)}{\Delta_q}$ is defined as*

$$\underset{(p_1,p_2)}{\Delta_q} u(k_1,k_2,k_3) = u(q^2k_1, q^2k_2, q^2k_3) - p_1 u(qk_1, qk_2, qk_3) - p_2 u(k_1,k_2,k_3) \tag{3}$$

and its inverse, denoted by $\underset{(p_1,p_2)}{\Delta_q^{-1}}$, is defined as below:

$$\text{if } \underset{(p_1,p_2)}{\Delta_q} v(k_1,k_2,k_3) = u(k_1,k_2,k_3), \text{ then } v(k_1,k_2,k_3) = \underset{(p_1,p_2)}{\Delta_q^{-1}} u(k_1,k_2,k_3). \tag{4}$$

Lemma 2.2. *If $q^{2n} - p_1 q^n - p_2 \neq 0$ for $n = 0, 1, 2, \cdots$, then*

$$\underset{(p_1,p_2)}{\Delta_q^{-1}} (k_1k_2k_3)^n = \frac{(k_1k_2k_3)^n}{q^{2n} - p_1 q^n - p_2} \text{ and } \underset{(p_1,p_2)}{\Delta_q^{-1}} (1) = \frac{1}{1 - p_1 - p_2}. \tag{5}$$

Proof. Replacing $u(k_1,k_2,k_3)$ by $(k_1k_2k_3)^n$ in (3), we find

$$\underset{(p_1,p_2)}{\Delta_q}(k_1k_2k_3)^n = (q^2(k_1k_2k_3))^n - p_1(q^n(k_1k_2k_3)^n) - p_2(k_1k_2k_3)^n$$

$$(k_1k_2k_3)^n = (q^{2n} - p_1q^n - p_2)\underset{(p_1,p_2)}{\Delta_q^{-1}}(k_1k_2k_3)^n \tag{6}$$

Again replacing $u(k_1,k_2,k_3)$ by $(k_1k_2k_3)^0$ in (3),

$$\underset{(p_1,p_2)}{\Delta_q}(k_1k_2k_3)^0 = (q^2(k_1k_2k_3))^0 - p_1(q^0(k_1k_2k_3)^0) - p_2(k_1k_2k_3)^0$$

$$\underset{(p_1,p_2)}{\Delta_q}(1) = (1-p_1-p_2)(1) \tag{7}$$

Hence (6) and (7) yield (5). □

Lemma 2.3. *Let $(k_1,k_2,k_3) \in \Re^3$ and $1-p_1-p_2 \neq 0$. Then we have*

$$\underset{(p_1,p_2)}{\Delta_q^{-1}}\log(k_1k_2k_3) = \frac{\log(k_1k_2k_3)}{1-p_1-p_2} - \frac{(2-p_1)\log q}{(1-p_1-p_2)^2}. \tag{8}$$

Proof. From (3), replacing $u(k_1,k_2,k_3)$ by $\log(k_1k_2k_3)$, we get

$$\underset{(p_1,p_2)}{\Delta_q}\log(k_1k_2k_3) = 2\log q + \log(k_1k_2k_3) - p_1\log q - p_1\log(k_1k_2k_3) - p_2\log(k_1k_2k_3)$$

$$\underset{(p_1,p_2)}{\Delta_q}\log k_1k_2k_3 = (2-p_1)\log q + (1-p_1-p_2)\log(k_1k_2k_3).$$

Applying $\underset{(p_1,p_2)}{\Delta_q^{-1}}$ on both sides

$$\underset{(p_1,p_2)}{\Delta_q^{-1}}\left[\underset{(p_1,p_2)}{\Delta_q}\log(k_1k_2k_3)\right] = \underset{(p_1,p_2)}{\Delta_q^{-1}}((2-p_1)\log q) + \underset{(p_1,p_2)}{\Delta_q^{-1}}(1-p_1-p_2)\log(k_1k_2k_3)$$

$$\log(k_1k_2k_3) = \underset{(p_1,p_2)}{\Delta_q^{-1}}\log(k_1k_2k_3)(2-p_1)\log q + \underset{(p_1,p_2)}{\Delta_q^{-1}}(1-p_1-p_2),$$

which gives (8). □

Lemma 2.4. *Let $(k_1,k_2,k_3) \in \Re^3$ and $q \neq 0$. Then we have*

$$\underset{(2\alpha,-\alpha^2)}{\Delta_q}u(k_1,k_2,k_3) = u(q^2k_1,q^2k_2,q^2k_3) - 2\alpha u(qk_1,qk_2,vk_3) + \alpha^2u(k_1,k_2,k_3).$$

Proof. The proof completes by putting $p_1 = 2\alpha$ and $p_2 = -\alpha^2$ in (3). □

3. Fibonacci sequence from three-dimensional q-difference equation

In this section, we introduce three dimensional Fibonacci sequence and its sum.

Definition 3.1. *For each pair $(p_1, p_2) \in R^2$, the three-dimensional Fibonacci sequence is defined as*

$$F_{(p_1,p_2)} = \Big\{F_n\Big\}_{n=0}^{\infty}, \tag{9}$$

where $F_0 = 1, F_1 = p_1$ and $F_n = p_1F_{n-1} + p_2F_{n-2}$ for $n \geq 2$.
When $p_1 = p_2 = 1$, (9) becomes the usual Fibonacci sequence.

Theorem 3.2. *[Three-Dimensional Finite q-Series] Let $F_n \in F_{(p_1,p_2)}$ and $(k_1, k_2, k_3) \in \Re^3$. Then we have*

$$\sum_{d=0}^{m} F_d u\Big(\frac{k_1}{q^{d+2}}, \frac{k_2}{q^{d+2}}, \frac{k_3}{q^{d+2}}\Big) = \underset{(p_1,p_2)}{\Delta_q^{-1}} u(k_1, k_2, k_3) - F_{m+1} \underset{(p_1,p_2)}{\Delta_q^{-1}} u\Big(\frac{k_1}{q^{m+1}}, \frac{k_2}{q^{m+1}}, \frac{k_3}{q^{m+1}}\Big)$$

$$- p_2F_m \underset{(p_1,p_2)}{\Delta_q^{-1}} u\Big(\frac{k_1}{q^{m+2}}, \frac{k_2}{q^{m+2}}, \frac{k_3}{q^{m+2}}\Big). \tag{10}$$

Proof. Taking $\underset{(p_1,p_2)}{\Delta_q^{-1}} u(k_1, k_2, k_3) = v(k_1, k_2, k_3)$

$\underset{(p_1,p_2)}{\Delta_q} v(k_1, k_2, k_3) = u(k_1, k_2, k_3)$

By (3), we write

$$v(q^2k_1, q^2k_2, q^2k_3) = u(k_1, k_2, k_3) + p_1v(qk_1, qk_2, qk_3) + p_2v(k_1, k_2, k_3) \tag{11}$$

Replacing (k_1, k_2, k_3) by $\Big(\frac{k_1}{q}, \frac{k_2}{q}, \frac{k_3}{q}\Big)$ in (11), we obtain

$$v(qk_1, qk_2, qk_3) = u\Big(\frac{k_1}{q}, \frac{k_2}{q}, \frac{k_3}{q}\Big) + p_1v(k_1, k_2, k_3) + p_2v\Big(\frac{k_1}{q}, \frac{k_2}{q}, \frac{k_3}{q}\Big) \tag{12}$$

Substituting the value of $v(qk_1, qk_2, qk_3)$ in (11), we get

$$v(q^2k_1, q^2k_2, q^2k_3) = u(k_1, k_2, k_3) + p_1\Big[v\left(\frac{k_1}{q}\frac{k_2}{q}\frac{k_3}{q}\right)$$

$$+p_1v(k_1, k_2, k_3) + p_2\left(\frac{k_1}{q}\frac{k_2}{q}\frac{k_3}{q}\right)\Big] + p_2v(k_1, k_2, k_3)$$

$$v(q^2k_1,q^2k_2,q^2k_3)=u(k_1,k_2,k_3)+p_1u\left(\frac{k_1}{q},\frac{k_2}{q},\frac{k_3}{q}\right)$$
$$+(p_1^2+p_2)v(k_1,k_2,k_3)+p_1p_2v\left(\frac{k_1}{q},\frac{k_2}{q},\frac{k_3}{q}\right). \quad (13)$$

Replace k_1,k_2,k_3 by $\left(\frac{k_1}{q},\frac{k_2}{q},\frac{k_3}{q}\right)$ in (12) and putting the value of $v(k_1,k_2,k_3)$ in (13), we obtain

$$v(q^2k_1,q^2k_2,q^2k_3)=u(k_1,k_2,k_3)+p_1u\left(\frac{k_1}{q},\frac{k_2}{q},\frac{k_3}{q}\right)+(p_1^2+p_2)\Big[u\left(\frac{k_1}{q^2},\frac{k_2}{q^2},\frac{k_3}{q^2}\right)$$
$$+p_1v\left(\frac{k_1}{q},\frac{k_2}{q},\frac{k_3}{q}\right)+p_2v\left(\frac{k_1}{q^2},\frac{k_2}{q^2},\frac{k_3}{q^2}\right)\Big]+p_1p_2v\left(\frac{k_1}{q},\frac{k_2}{q},\frac{k_3}{q}\right)$$
$$+p_1(p_1^2+p_2)v\left(\frac{k_1}{q},\frac{k_2}{q},\frac{k_3}{q}\right)+p_2(p_1^2+p_2)v\left(\frac{k_1}{q^2},\frac{k_2}{q^2},\frac{k_3}{q^2}\right)+p_1p_2v\left(\frac{k_1}{q},\frac{k_2}{q},\frac{k_3}{q}\right)$$
$$v(q^2k_1,q^2k_2,q^2k_3)=u(k_1,k_2,k_3)+p_1u\left(\frac{k_1}{q},\frac{k_2}{q},\frac{k_3}{q}\right)+(p_1^2+p_2)u\left(\frac{k_1}{q^2},\frac{k_2}{q^2},\frac{k_3}{q^2}\right)$$
$$+\Big\{p_1(p_1^2+p_2)+p_1p_2\Big\}v\left(\frac{k_1}{q},\frac{k_2}{q},\frac{k_3}{q}\right)+p_2(p_1^2+p_2)v\left(\frac{k_1}{q},\frac{k_2}{q},\frac{k_3}{q}\right). \quad (14)$$

Since $F_n\in F_{(p_1,p_2)}$ we get

$$v(q^2k_1,q^2k_2,q^2k_3)=F_0u(k_1,k_2,k_3)+F_1u\left(\frac{k_1}{q},\frac{k_2}{q},\frac{k_3}{q}\right)+F_2u\left(\frac{k_1}{q^2},\frac{k_2}{q^2},\frac{k_3}{q^2}\right)$$
$$+F_3v\left(\frac{k_1}{q},\frac{k_2}{q},\frac{k_3}{q}\right)+p_2F_2v\left(\frac{k_1}{q^2},\frac{k_2}{q^2},\frac{k_3}{q^2}\right)$$

Proceeding like this, we arrive

$$v(q^2k_1,q^2k_2,q^2k_3)=F_0u(k_1,k_2,k_3)+F_1u\left(\frac{k_1}{q},\frac{k_2}{q},\frac{k_3}{q}\right)+\cdots+F_mu\Big(\frac{k_1}{q^m},\frac{k_2}{q^m},\frac{k_3}{q^m}\Big)$$
$$+F_{m+1}v\Big(\frac{k_1}{q^{m-1}},\frac{k_2}{q^{m-1}},\frac{k_3}{q^{m-1}}\Big)+p_2F_mv\Big(\frac{k_1}{q^m},\frac{k_2}{q^m},\frac{k_3}{q^m}\Big). \quad (15)$$

Now replace (k_1,k_2,k_3) by $\left(\frac{k_1}{q^2},\frac{k_2}{q^2},\frac{k_3}{q^2}\right)$ in (15) we get

$$v(k_1,k_2,k_3)=F_0u\left(\frac{k_1}{q^2},\frac{k_2}{q^2},\frac{k_3}{q^2}\right)+F_1u\left(\frac{k_1}{q^3},\frac{k_2}{q^3},\frac{k_3}{q^3}\right)+\cdots+F_mu\left(\frac{k_1}{q^{m+2}},\frac{k_2}{q^{m+2}},\frac{k_3}{q^{m+2}}\right)$$
$$+F_{m+1}v\left(\frac{k_1}{q^{m+1}},\frac{k_2}{q^{m+1}},\frac{k_3}{q^{m+1}}\right)+p_2F_mv\left(\frac{k_1}{q^{m+2}},\frac{k_2}{q^{m+2}},\frac{k_3}{q^{m+2}}\right)$$

$$\underset{(p_1,p_2)}{\Delta_q^{-1}} u(k_1,k_2,k_3) - F_{m+1} \underset{(p_1,p_2)}{\Delta_q^{-1}} u\left(\frac{k_1}{q^{m+1}},\frac{k_2}{q^{m+1}},\frac{k_3}{q^{m+1}}\right)$$
$$-p_2F_m \underset{(p_1,p_2)}{\Delta_q^{-1}} u\left(\frac{k_1}{q^{m+2}},\frac{k_2}{q^{m+2}},\frac{k_3}{q^{m+2}}\right) = \sum_{d=0}^{m} F_d u\left(\frac{k_1}{q^{d+2}},\frac{k_2}{q^{d+2}},\frac{k_3}{q^{d+2}}\right),$$

which completes the proof of the theorem. □

Example 3.3. *Considering* $p_1 = 2$, $p_2 = 4$, $m = 3$, $n = 2$, $q = 5$, $k_1 = 1$, $k_2 = 3$, $k_3 = 2$, $u(k_1,k_2,k_3) = (k_1k_2k_3)^n$, *then we are able to find*
$F_0 = 1$, $F_1 = p_1 = 2$, $F_2 = 8$, $F_3 = 24$, *and* $F_4 = 80$.
From (10), $L.H.S = \sum_{d=0}^{3} F_d u\left(\frac{k_1}{q^{d+2}},\frac{k_2}{q^{d+2}},\frac{k_3}{q^{d+2}}\right) = \sum_{d=0}^{3} F_d \left(\frac{k_1}{q^{d+2}}.\frac{k_2}{q^{d+2}}.\frac{k_3}{q^{d+2}}\right)^n$

$$= 0.063033753$$

Now,
$$\underset{(p_1,p_2)}{\Delta_q^{-1}} u(k_1,k_2,k_3) = \underset{(p_1,p_2)}{\Delta_q^{-1}} (k_1k_2k_3)^n$$
$$= \frac{(k_1k_2k_3)^n}{q^{2n} - p_1q^n - p_2} = \frac{(1*3*2)^2}{5^4 - (2*5)^2 - 4}$$
$$= 0.063047285$$

$$\underset{(p_1,p_2)}{\Delta_q^{-1}} u\left(\frac{k_1}{q^{m+1}}.\frac{k_2}{q^{m+1}}.\frac{k_3}{q^{m+1}}\right)^n = \frac{\left(\frac{k_1}{q^{m+1}}.\frac{k_2}{q^{m+1}}.\frac{k_3}{q^{m+1}}\right)^n}{q^{2n} - p_1q^n - p_2}$$
$$= \frac{\left(\frac{6}{5^4}\right)^2}{571} = 1.614010508 \times 10^{-7}$$
$$\underset{(p_1,p_2)}{\Delta_q^{-1}} u\left(\frac{k_1}{q^{m+2}}.\frac{k_2}{q^{m+2}}.\frac{k_3}{q^{m+2}}\right)^n = \frac{\left(\frac{6}{5^5}\right)^2}{571} = 6.456042032 \times 10^{-9}$$

$R.H.S = 0.063047285 - F_4(1.614010508 \times 10^{-7}) - (4)F_3(6.456042032 \times 10^{-9})$
$= 0.063033753$
Hence the theorem is verified.

The following corollary is a special case of Theorem 3.2.

Corollary 3.4. *Assume that $p_1 + p_2 \neq 1$ and $F_n \in F_{(p_1,p_2)}$. Then we have*

$$\sum_{d=0}^{m} F_d = \frac{1 - F_{m+1} - p_2 F_m}{1 - p_1 - p_2}. \tag{16}$$

Proof. From (10) replacing $u(k)$ by k^0, we find

$$\sum_{d=0}^{m} F_d \Big(\frac{k_1}{q^{d+2}}.\frac{k_2}{q^{d+2}}.\frac{k_3}{q^{d+2}}\Big)^0 = \underset{(p_1,p_2)}{\Delta_q^{-1}} (k_1k_2k_3)^0$$

$$-F_{m+1} \underset{(p_1,p_2)}{\Delta_q^{-1}} \left(\frac{k_1}{q^{m+1}}, \frac{k_2}{q^{m+1}}, \frac{k_3}{q^{m+1}}\right)^0 - p_2F_m \underset{(p_1,p_2)}{\Delta_q^{-1}} \left(\frac{k_1}{q^{m+2}}, \frac{k_2}{q^{m+2}}, \frac{k_3}{q^{m+2}}\right)^0$$

$$\sum_{d=0}^{m} F_d = \left(\underset{(p_1,p_2)}{\Delta_q^{-1}} - F_{m+1} \underset{(p_1,p_2)}{\Delta_q^{-1}} - p_2F_m \underset{(p_1,p_2)}{\Delta_q^{-1}} \right) (1),$$

which yields (16). $\square$

An example is given below to illustrate Corollary 3.4 :

Example 3.5. *Considering $m = 5$, $p_1 = 10$, $p_2 = 15$, then from (16) we get $F_0 = 1$, $F_1 = p_1 = 10$, $F_2 = 115$, $F_3 = 1300$, $F_4 = 14725$, $F_5 = 166750$ and $F_6 = 1888375$. Now $L.H.S = \sum_{d=0}^{m} F_d = 182901$ and*

$$R.H.S = \frac{1 - F_{m+1} - p_2F_m}{1 - p_1 - p_2} = \frac{1 - F_6 - p_2F_5}{1 - p_1 - p_2} = 182901.$$

4. Product formula of three dimensional q-difference equation

Theorem 4.1. *For the real valued functions $u(k_1, k_2, k_3)$ and $v(k_1, k_2, k_3)$, we have*

$$\underset{(p_1,p_2)}{\Delta_q^{-1}} \big(u(k_1, k_2, k_3)v(k_1, k_2, k_3)\big) = \frac{1}{p_2}\Big\{u(k_1, k_2, k_3) \underset{(0,1)}{\Delta_q^{-1}} v(k_1, k_2, k_3)$$

$$- \underset{(p_1,p_2)}{\Delta_q^{-1}} \Big(\underset{(p_1,p_2)}{\Delta_q} u(k_1, k_2, k_3) \underset{(0,1)}{\Delta_q^{-1}} v(q^2k_1, q^2k_2, q^2k_3)\Big)$$

$$- p_1 \underset{(p_1,p_2)}{\Delta_q^{-1}} \Big(u(qk_1, qk_2, qk_3) \underset{(1,0)}{\Delta_q} \big(\underset{(0,1)}{\Delta_q^{-1}} v(k_1, k_2, k_3)\big)\Big)\Big\}. \tag{17}$$

Proof. From (1), we find that

$$\underset{(p_1,p_2)}{\Delta_q} u(k_1, k_2, k_3) = u(q^2k_1, q^2k_2, q^2k_3) - p_1u(qk_1, qk_2, qk_3) - p_2u(k_1, k_2, k_3)$$

$$\underset{(p_1,p_2)}{\Delta_q} u(k_1, k_2, k_3)w(k_1, k_2, k_3) = u(q^2k_1, q^2k_2, q^2k_3)w(q^2k_1, q^2k_2, q^2k_3)$$

$$-p_1u(qk_1, qk_2, qk_3)w(qk_1, qk_2, qk_3) - p_2u(k_1, k_2, k_3)w(k_1, k_2, k_3)$$

$$u(k_1,k_2,k_3)w(k_1,k_2,k_3) = \Delta_q^{-1}\Big[w(q^2k_1,q^2k_2,q^2k_3)[\Delta_q\, u(k_1,k_2,k_3)]$$
$$+p_1u(qk_1,qk_2,qk_3)\,\Delta_q(\Delta_q^{-1}\,w(k_1,k_2,k_3)) + p_2u(k_1,k_2,k_3)v(k_1,k_2,k_3)\Big]$$
$$u(k_1,k_2,k_3)\,\Delta_q^{-1}\,v(k_1,k_2,k_3) = \Delta_q^{-1}\Big[\,\Delta_q^{-1}\,v(q^2k_1,q^2k_2,q^2k_3)[\Delta_q\,u(k_1,k_2,k_3)]$$
$$+p_1u(qk_1,qk_2,qk_3)\,\Delta_q(\Delta_q^{-1}\,w(k_1,k_2,k_3)) + p_2u(k_1,k_2,k_3)v(k_1,k_2,k_3)\Big]$$
$$u(k_1,k_2,k_3)\,\Delta_q^{-1}\,v(k_1,k_2,k_3) = \Delta_q^{-1}\Big[\,\Delta_q^{-1}\,v(q^2k_1,q^2k_2,q^2k_3)[\Delta_q\,u(k_1,k_2,k_3)]$$
$$+p_1u(qk_1,qk_2,qk_3)\,\Delta_q(\Delta_q^{-1}\,w(k_1,k_2,k_3)) + p_2u(k_1,k_2,k_3)v(k_1,k_2,k_3)\Big]$$
$$u(k_1,k_2,k_3)\,\Delta_q^{-1}\,v(k_1,k_2,k_3) = \Delta_q^{-1}\Big[\,\Delta_q\,u(k_1,k_2,k_3)\,\Delta_q^{-1}\,v(q^2k_1,q^2k_2,q^2k_3)$$
$$+p_1\,\Delta_q^{-1}\Big(u(qk_1,qk_2,qk_3)\,\Delta_q(\Delta_q^{-1}\,v(k_1,k_2,k_3))\Big) + p_2\,\Delta_q^{-1}\,u(k_1,k_2,k_3)v(k_1,k_2,k_3)\Big]$$
$$-p_2\,\Delta_q^{-1}\Big(u(k_1,k_2,k_3)v(k_1,k_2,k_3)\Big) = -u(k_1,k_2,k_3)\,\Delta_q^{-1}\,v(k_1,k_2,k_3)$$
$$+\,\Delta_q^{-1}\Big(\,\Delta_q\,u(k_1,k_2,k_3)\,\Delta_q^{-1}\,v(q^2k_1,q^2k_2,q^2k_3)\Big)$$
$$+p_1\,\Delta_q^{-1}\Big(u(qk_1,qk_2,qk_3)\,\Delta_q(\Delta_q^{-1}\,v(k_1,k_2,k_3))\Big),$$

which completes the proof. □

Corollary 4.2. *For real valued function $v(k_1,k_2,k_3)$ and for $k_1,k_2,k_3 > 0$, we have*

$$\Delta_q^{-1}\Big(v(k_1,k_2,k_3)\log(k_1k_2k_3)\Big) = \frac{1}{p_2}\Big[\log(k_1k_2k_3)\,\Delta_q^{-1}\,v(k_1,k_2,k_3)$$
$$-\,\Delta_q^{-1}\Big(\,\Delta_q\log(k_1k_2k_3)\,\Delta_q^{-1}\,v(q^2k_1,q^2k_2,q^2k_3)\Big)$$
$$-\,p_1\,\Delta_q^{-1}\Big(\,\log(qk_1,qk_2,qk_3)\,\Delta_q(\Delta_q^{-1}\,v(k_1,k_2,k_3))\Big)\Big] \quad (18)$$

Proof. The proof is trivial by replacing $u(k_1,k_2,k_3)$ by $\log(k_1k_2k_3)$ in (17). □

Corollary 4.3. *For $q,k_1,k_2,k_3 > 0$, $1-p_1q-p_2q^2 \neq 0$ and $F_n \in F_{(p_1,p_2)}$, we have*

$$\underset{(p_1,p_2)}{\Delta_q^{-1}}\Big(\frac{1}{k_1k_2k_3}\log(k_1k_2k_3)\Big) = \frac{q^2}{(1-p_1q-p_2q^2)k_1k_2k_3}$$
$$\Big\{\log(k_1k_2k_3) - \frac{(2-p_1q)\log(q)}{(1-p_1q-p_2q^2)}\Big\} \quad (19)$$

and hence

$$(1-F_{m+1}q^{m+1}-a_2F_mq^{m+2})\,\underset{(p_1,p_2)}{\Delta_q^{-1}}\Big(\frac{1}{k_1k_2k_3}\log k_1k_2k_3\Big)$$

$$+\Big((m+1)F_{(m+1)}q^{m+1} + p_2(m+2)F_mq^{m+2}\Big)$$
$$\times \frac{q^2 \log q}{(1-p_1q-p_2q^2)k_1k_2k_3} = \sum_{d=0}^{m} F_d \frac{q^{d+2}}{k_1k_2k_3} \log\Big(\frac{k_1k_2k_3}{q^{d+2}}\Big) \qquad (20)$$

Proof. Taking $v(k_1,k_2,k_3) = \dfrac{1}{k_1k_2k_3}$ in (18), we get

$$\underset{(p_1,p_2)}{\Delta_q^{-1}}\Big(\frac{1}{k_1k_2k_3}\log(k_1k_2k_3)\Big) = \frac{1}{p_2}\Big\{ \log(k_1k_2k_3)\underset{(0,1)}{\Delta_q^{-1}}\frac{1}{k_1k_2k_3}$$
$$- \underset{(p_1,p_2)}{\Delta_q^{-1}}\Big(\underset{(p_1,p_2)}{\Delta_q} \log(k_1k_2k_3)\underset{(0,1)}{\Delta_q^{-1}}\frac{1}{q^2(k_1k_2k_3)}\Big)$$
$$- p_1 \underset{(p_1,p_2)}{\Delta_q^{-1}}\Big(\log(q(k_1k_2k_3))\underset{(1,0)}{\Delta_q}(\underset{(0,1)}{\Delta_q^{-1}}\frac{1}{k_1k_2k_3})\Big)\Big\}$$

$$\underset{(p_1,p_2)}{\Delta_q^{-1}}\Big(\frac{1}{k_1k_2k_3}\log(k_1k_2k_3)\Big) = \frac{1}{p_2}\Big(\frac{q^2}{1-q^2}\Big)\Big\{\log(k_1k_2k_3)\frac{1}{k_1k_2k_3}$$
$$-\frac{1}{q^2}\underset{(p_1,p_2)}{\Delta_q^{-1}}\Big((2-p_1)\log q\frac{1}{k_1k_2k_3} + (1-p_1-p_2)\log(k_1k_2k_3)\frac{1}{k_1k_2k_3}\Big)$$
$$-p_1\Big(\frac{1-q}{q^2}\Big)\underset{(p_1,p_2)}{\Delta_q^{-1}}\Big(\log(q(k_1k_2k_3))\frac{1}{k_1k_2k_3}\Big)\Big)\Big\}$$

$$\Big[1+\Big(\frac{1}{p_2}\Big)\Big(\frac{q^2}{1-q^2}\Big)\frac{1}{q^2}(1-p_1-p_2) + (\frac{1}{p_2})(\frac{q^2}{1-q^2})p_1\frac{1-q}{q^2}\Big]\underset{(p_1,p_2)}{\Delta_q^{-1}}\Big(\frac{1}{k_1k_2k_3}\log(k_1k_2k_3)\Big)$$
$$= \Big(\frac{1}{p_2}\Big)\Big(\frac{q^2}{1-q^2}\Big)\frac{1}{k_1k_2k_3}\Big\{\log(k_1k_2k_3) - (2-p_1)\log q\Big(\frac{q^2}{1-p_1q-p_2q^2}\Big)$$
$$-p_1\Big(\frac{1-q}{q^2}\Big)\log q\Big(\frac{q^2}{1-p_1q-p_2q^2}\Big)\Big\}$$

$$\underset{(p_1,p_2)}{\Delta_q^{-1}}\Big(\frac{1}{k_1k_2k_3}\log(k_1k_2k_3)\Big) = \frac{q^2}{1-p_1q-p_2q^2(k_1k_2k_3)}$$
$$\Big\{\log(k_1k_2k_3) - \Big(\frac{1}{1-p_1q-p_2q^2}\Big)\Big\{2\log q - p_1\log q + p_1\log q - qp_1\log q\Big\}\Big\},$$

which gives (19).

Again (20) is obvious by replacing $u(k_1,k_2,k_3)$ by $\dfrac{1}{k_1k_2k_3}(\log(k_1k_2k_3))$ in (10). $\square$

The numerical verifications for Fibonacci series using real valued functions is given below:

Example 4.4. *Considering* $m = 2$, $k_1 = -10$, $k_2 = -15$, $k_3 = -19$, $q = -20$, $p_1 = 2$, $p_2 = 4$, *then we find* $F_0 = 1$, $F_1 = p_1 = 2$, $F_2 = 8$, $F_3 = 24$ *and* $F_4 = 80$.

$$L.H.S = F_0\Big(\frac{(-20)^2}{-2850}\Big)\log\Big(\frac{-2850}{(-20)^2}\Big) + F_1\Big(\frac{(-20)^3}{-2850}\Big)\log\Big(\frac{-2850}{(-20)^3}\Big) + F_2\Big(\frac{(-20)^4}{-2850}\Big)\log\Big(\frac{-2850}{(-20)^4}\Big)$$

$$= -788.0361283 \quad \textit{and}$$

$$R.H.S = (1 - 24((-20)^3) - 4(8)((-20)^4))\Big\{\frac{(-20)^2}{(-1559)(-2850)} \Big\{(-3.45484486) - 0.03505098\Big\} + 19904000(-1.171268128 \times 10^{-4})\Big\}$$
$$+\Big((3)(72)(-20)^3 - (5)(4)(13)(-20)^4\Big)\Big[\frac{(-20)^2(1.041392685)}{(-648)(-2850)}\Big]$$
$$= -788.0361285.$$

Hence the theorem is verified.

5. Conclusion

In this research work, the summation solution and the closed form solutions of polynomials and logarithmic functions using three dimensional q-difference equation have been obtained. Also several results using three dimensional q-difference operator are derived. Moreover generalized product formula for polynomial, reciprocal of polynomial and logarithmic functions using inverse of three dimensional q-difference operator is obtained. Consequently, relevant examples are being given to investigate the results.

References

[1] Britto Antony Xavier .G, Chandrasekar .V, and Suresh .K, *Theory and Applications of Generalized q-derivative operator*, International Conference On Mathematical Computer Engineering - ICMCE - 2013.

[2] Britto Antony Xavier .G, Gerly .T.G, and Nasira Begum .H, *Finite series of polynomials and polynomial factorials arising from generalised q-difference operator*, Far East Journal of Mathematical Sciences, 94(1) (2014), 47-63.

[3] Jerzy Popenda and Blazej Szmanda, *On the Oscillation of Solutions of Certain Difference Equations*, Demonstratio Mathematica, XVII(1) (1984), 153-164.

[4] Britto Antony Xavier .G, Gerly .T.G, and Vasantha Kumar S.U, *Multi-Series Solution of Generalized q-alpha Difference Equation*, International Journal of Applied Engineering Research, 10(72) (2015), 97-101.

[5] Carmichael, *RD: The general theory of linear q-difference equations*, Am. J. Math., 34 (1912), 147-168.

[6] Jackson, *FH: On q-difference equations*, Am. J. Math., 32 (1910), 305-314.

[7] Maria Susai Manuel .M, Britto Antony Xavier .G, and Thandapani .E, *Theory of Generalized Difference Operator and Its Applications*, Far East Journal of Mathematical Sciences, 20(2) (2006), 163 - 171.

[8] Maria Susai Manuel .M, Brito Antony Xaviar .G, Chandrasekar .V, and Pugalarasu .R, *Theory and Application of the Generalized Derivative Opertor of the n^{th} kind (Part I)*, Demonstration Mathematical, 45(1) (2012), 95 - 106.