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A Complex Variable Method for Secondary Flow in A Slowly Rotating Straight Elliptic Pipe

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Abstract. A complex variable method is proposed in this paper to find the closed form expressions for non-dimensional stream function and velocity component in the primary and secondary flows of a viscous incompressible fluid through a straight elliptic pipe rotating about an axis through any diameter. This method works successfully whenever the cross-section of the pipe is expressed in the form $Z\bar{Z} = f(Z) + f(\bar{Z})$, where $Z = x + iy$ and $f(\bar{Z})$ is the complex conjugate of $f(Z)$, x and y being rectangular Cartesian co-ordinates in a cross-section of the pipe. The streamline pattern for the secondary flow is discussed in detail along and near the edge of rotating pipe.

Key words: Secondary flow, rotation, Reynolds number, complex conjugate.

1. Introduction and Preliminaries

The problem of viscous incompressible fluid flow through rotating straight pipes has been a subject of intensive and extensive study for many years not only for their theoretical interest but also for their possible applications in industry and other fields. The streamline motion of viscous incompressible fluid through a straight pipe of circular cross-section rotating with a constant angular velocity about a line perpendicular to the pipe axis was studied earlier by Barua [1] and Benton [2]. A secondary flow in planes perpendicular to the pipe length has been observed in these investigations. The effect of this secondary flow is a reduction of the flow rate corresponding to a given pressure gradient.

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This observation is qualitatively similar to that observed by Dean [3] for stationary curved pipe. Vidyannidhi and Nigam [5] have obtained the theoretical solution for the secondary flow in the simpler geometry of a channel. The formation of the secondary flow in slowly rotating straight elliptic pipe has been noted by Lakshmana Rao and Bhjanga Rao [4]. Here the solution is obtained by perturbation on the classical Hagen-Poiseuille flow.

A fundamental understanding of fluid flow is essential to almost every industry related with chemical engineering. In the chemical and manufacturing industries, large flow networks are necessary to achieve continuous transport of products and raw materials from different processing units. This requires a detailed understanding of fluid flow in pipes. Energy input to the gas or liquid is needed to make it flow through the pipe. In view of these applications Berman et al [6], Lie and Hsu [9], Barua [1], Lie et al [10], Petrakis [7], Ravi K Sharma et al [8] have examined the different problems on flow through straight and curved pipes.

The aim of this note is to demonstrate the application of a complex variable method for the determination of the primary and secondary flows if a viscous incompressible fluid through a straight elliptic pipe rotating slowly about an axis perpendicular to its length. In fact, this method works successfully whenever the cross-section of the pipe is expressed in the form $z\bar{z} = f(z) + f(\bar{z})$, where $z = x + iy$ and $f(\bar{z})$ is the complex conjugate of $f(z)$, x and y being rectangular Cartesian co-coordinates in a cross of the pipe.

2. Mathematical Formulation

Consider the elliptic pipe whose cross-section is given by

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1 \quad (a > b > 0) \tag{1}$$

With the Z - axis along the axis of the pipe (i.e., OZ) and X, Y axes are in the plane perpendicular to Y - axis which rotates with uniform angular speed Ω about the diameter of the cross-section inclined at an angle α with the major axis $\left(0 \leq \alpha \leq \frac{\pi}{2}\right)$. The equations of steady flow and of continuity [4] are

$$\rho\left(V.\nabla\right)V + 2\rho\left(\Omega \times V\right) + \rho\Omega \times \left(\Omega \times r\right) = -\nabla P + \rho v \nabla^2 V \tag{2}$$

$$\nabla . V = 0 \tag{3}$$

where $V = (U, V, W)$. The above equation under the assumption that motion is uniform in every cross-section of the pipe reduce to

$$U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} + 2\Omega W \sin\alpha = -\frac{\partial P}{\partial X} + v\nabla^2 U \tag{4}$$

$$U\frac{\partial V}{\partial X} + V\frac{\partial V}{\partial Y} - 2\Omega W \cos\alpha = -\frac{\partial P}{\partial Y} + v\nabla^2 V \tag{5}$$

$$U\frac{\partial W}{\partial X} + V\frac{\partial W}{\partial Y} - 2\Omega(V\cos\alpha - U\sin\alpha) = -\frac{\partial P}{\partial Z} + v\nabla^2 W \tag{6}$$

$$\frac{\partial U}{\partial X} + \frac{\partial U}{\partial Y} = 0 \tag{7}$$

Where

$$P = \frac{p}{\rho} - \frac{1}{2}\Omega^2 \left[(Y\cos\alpha - X\sin\alpha)^2 + Z\right] \tag{8}$$

The equation (7) will be satisfied by taking the velocity components U and V as

$$U = -\frac{\partial\psi}{\partial y}, V = \frac{\partial\psi}{\partial X} \tag{9}$$

where ψ is the stream function. The basic equation (4), (5), (6) after eliminating P from equation (4) and (5) reduce to

$$-\frac{\partial\left(\psi, \nabla^2\psi\right)}{\partial\left(X, Y\right)} + 2\Omega\left(\frac{\partial W}{\partial X}\cos\alpha + \frac{\partial W}{\partial Y}\sin\alpha\right) = -v\nabla^2\nabla^2\psi \tag{10}$$

and

$$-\frac{\partial\left(W, \psi\right)}{\partial\left(X, Y\right)} + 2\Omega\left(\frac{\partial\psi}{\partial X}\cos\alpha + \frac{\partial\psi}{\partial Y}\sin\alpha\right) = -\frac{\partial P}{\partial Z} + v\nabla^2 W \tag{11}$$

And on the boundary Γ of the pipe cross-section we have no slip condition viz.,

$$U = V = W = 0 \tag{12}$$

Introducing the non-dimensional quantities

$$\psi = v\bar{\psi}, W = \frac{vw}{a}, X = ax, Y = ay, R = \frac{2\Omega a^2}{v} \tag{13}$$

where $\Omega = |\Omega|$ is the angular speed of the rotating pipe and R the rotation Reynolds number. The slowness of the flow is characterized by the smallness of R. Now the equation (10) and (11) take the form

$$-\frac{\partial\left(\bar{\psi}, \nabla^2\bar{\psi}\right)}{\partial\left(x, y\right)} + R\left(\frac{\partial W}{\partial x}\cos\alpha + \frac{\partial W}{\partial y}\sin\alpha\right) = \nabla^2\nabla^2\bar{\psi} \tag{14}$$

and

$$-\frac{\partial\left(\bar{\psi}, w\right)}{\partial\left(x, y\right)}+R\left(\frac{\partial\bar{\psi}}{\partial x}\cos\alpha+\frac{\partial\bar{\psi}}{\partial y}\sin\alpha\right)=\frac{a^3}{v^2}\frac{\partial P}{\partial Z}\nabla^2 w \qquad (15)$$

where

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} \qquad (16)$$

Then equation (2.15) can be written as

$$\frac{\partial\left(\bar{\psi}, w\right)}{\partial\left(x, y\right)}+R\left(\frac{\partial\bar{\psi}}{\partial x}\cos\alpha+\frac{\partial\bar{\psi}}{\partial y}\sin\alpha\right)=c+\nabla^2 w \qquad (17)$$

where

$$c=\frac{a^3}{v^2}\frac{\partial P}{\partial Z} \qquad (18)$$

is the parameter representing constant pressure gradient in the direction of the pipe length. we seek the solution of the above system, valid for small values of R, in the form

$$\omega\left(x, y\right)-\omega_0\left(x, y\right)+R\omega_1\left(x, y\right)+R^2\omega_2\left(x, y\right)+\cdots \qquad (19)$$

$$\bar{\psi}\left(x, y\right)=R\bar{\psi}_1\left(x, y\right)+R^2\bar{\psi}_2\left(x, y\right)+\cdots \qquad (20)$$

Substituting (19), (20) in (14) and comparing the coefficient of R^0 and R .We obtain the set of differential equation for ω_0, ψ_1 and ω_1 is

$$\nabla^2\omega_0=c \qquad (21)$$

$$\nabla^2\nabla^2\bar{\psi}_1=-\left(\cos\alpha\frac{\partial\omega_0}{\partial x}+\sin\alpha\frac{\partial\omega_0}{\partial y}\right) \qquad (22)$$

and The boundary conditions relevant to the product are that on Γ .

$$\omega_0=0, \frac{\partial\bar{\psi}_1}{\partial x}=\frac{\partial\bar{\psi}_1}{\partial y}=0, \omega_1=0 \qquad (23)$$

3. Complex variable Method

Introducing the complex variable $z = x + iy, \bar{z} = x - iy$ where $\bar{z}$ is conjugate of z. Then the equation (21) to (23) can be expressed in terms of complex quantities as

$$4\frac{\partial^2 \omega_0}{\partial z \partial \bar{z}} = c \tag{24}$$

$$16\frac{\partial^4 \bar{\psi}_1}{\partial z^2 \partial \bar{z}^2} = -\left[e^{i\alpha}\frac{\partial \omega_0}{\partial z} + e^{-i\alpha}\frac{\partial \omega_0}{\partial \bar{z}}\right] \tag{25}$$

and

$$4\frac{\partial^2 \omega_1}{\partial z \partial \bar{z}} = -2i\frac{\partial\left(\bar{\psi}_1, \omega_0\right)}{\partial\left(z, \bar{z}\right)} \tag{26}$$

with boundary conditions

$$\omega_0 = 0, \frac{\partial \bar{\psi}_1}{\partial z} = \frac{\partial \bar{\psi}_1}{\partial \bar{z}} = 0, \omega_1 = 0 on \Gamma \tag{27}$$

we seek a solution of the above equation whenever cross-section is expressed as

$$z\bar{z} = f\left(z\right) + f\left(\bar{z}\right) \tag{28}$$

The solution to equation (24) is given as

$$\omega_0 = \frac{c}{4}\left[z\bar{z} - f\left(z\right) + f\left(\bar{z}\right)\right] \tag{29}$$

where $f\left(z\right)$ is an analytic on and within Γ . Here the cross-section of the pipe is elliptic, i.e.,

$$\Gamma : x^2 + \frac{y^2}{k^2} = 1 \tag{30}$$

where $k = \dfrac{b}{a} = \sqrt{1 - e^2}$ and e is the eccentricity of the ellipse. Hence the elliptic cross-section can be written as

$$z\bar{z} = \left[\frac{\left(1 - k^2\right) z^2 + 2k^2}{2\left(k^2 + 1\right)}\right] + \left[\frac{\left(1 - k^2\right)^{-2} z + 2k^2}{2\left(k^2 + 1\right)}\right] \tag{31}$$

On identifying this with equation (28), we get

$$f\left(z\right) = \frac{\left(1 - k^2\right) z^2 + 2k^2}{2\left(k^2 + 1\right)} \qquad \text{and} \qquad f\left(\bar{z}\right) = \frac{\left(1 - k^2\right)^{-2} z + 2k^2}{2\left(k^2 + 1\right)} \tag{32}$$

Hence

$$\omega_0 = \frac{c}{4}\left[z\bar{z} - \frac{\left(1 - k^2\right) z^2 + 2k^2}{2\left(k^2 + 1\right)} - \frac{\left(1 - k^2\right)^{-2} z + 2k^2}{2\left(k^2 + 1\right)}\right] \tag{33}$$

$$= \frac{ck^2}{2(1 + k^2)}\left(x^2 + \frac{y^2}{k^2} - 1\right) \qquad (\text{ in Cartesian form }) \tag{34}$$

Substituting (33) in the equation (25) we get the equation for $\bar{\psi}$.

$$\frac{\partial^4 \bar{\psi}_1}{\partial z^2 \partial \bar{z}^{-2}} = - \frac{c}{64\left(1+k^2\right)}[\left\{\left(1+k^2\right)e^{i\alpha}+\left(1+k^2\right)e^{-i\alpha}\right\}z$$
$$+\left\{\left(1+k^2\right)e^{i\alpha}+\left(1+k^2\right)e^{-i\alpha}\right\}\bar{z}] \tag{35}$$

The solution of this equation, subject to the boundary conditions

$$\frac{\partial \bar{\psi}_1}{\partial z} = 0, \qquad \frac{\partial \bar{\psi}_1}{\partial \bar{z}} = 0 \qquad \text{or} \qquad \bar{\psi}_1 = 0 \tag{36}$$

can be expressed as

$$\bar{\psi}_1 = (Az + B\bar{z} + C)\left(z\bar{z} - D\left(z^2\_z^{-2}\right) - E\right)^2 \tag{37}$$

where

$$D = \frac{1-k^2}{2\left(k^2+1\right)} \quad \text{and} \quad E = \frac{4k^2}{2\left(k^2+1\right)} \tag{38}$$

The solution (37) is so structured that the boundary conditions (36) are satisfied. We have to obtain the coefficient A, B abd C such that equation (35) is satisfied. Now from (37), we get

$$\frac{\partial^4 \bar{\psi}_1}{\partial z^2 \partial z^{-2}} = 12[\left\{\left(1+2D^2\right)A - 2DB\right\}z + \left\{-2DA + B\left(1+2D^2\right)\right\}\bar{z} + \{4C\}\left(1+2D^2\right)] \tag{39}$$

Comparing this with equation (35), we notice that

$$A = T^*\left[\left(k^2-1\right)^3 e^{i\alpha} + \left(k^6+7k^4+7k^2+1\right)e^{-i\alpha}\right]$$
$$B = T^*\left[\left(k^6+7k^4+7k^2+1\right)e^{i\alpha} + \left(k^2-1\right)^3 e^{-i\alpha}\right] \tag{40}$$
$$= \bar{A}\,(= conjugate of A) \tag{41}$$

and

$$c = 0 \tag{42}$$

Hence

$$\bar{\psi}_1 = T^*\left(Az + \bar{A}z\right)\left(z\bar{z} - D\left(z^2+z^{-2}\right) - E\right)^2$$
$$\bar{\psi}_1 = T^*\left[\left\{\left(k^2-1\right)^3 e^{i\alpha} + \left(k^6+7k^4+7k^2+1\right)e^{-i\alpha}\right\}z + \text{conjugate}\right]$$
$$\left[z\bar{z} - \frac{(1-k)^2\left(z^2+z^{-2}\right)}{2c\left(k^2+1\right)} - \frac{4k^2}{2\left(k^2+1\right)}\right]^2 \tag{43}$$

where

$$T^* = \left\{\frac{-c\left(1+k^2\right)}{384\left(1+2k^2+5k^4\right)\left(5+2k^2+k^4\right)}\right\} \tag{44}$$

Now using $e^{\pm i\alpha} = \cos\alpha \pm i\sin\alpha$; $z = x + iy$ and $\bar{z} = x - iy$ we get in Cartesian form as

$$\bar{\psi}_1 = \frac{-ck^4}{24\left(1+k^2\right)}\left\{\frac{k^2\cos\alpha}{1+2k^2+5k^4}x + \frac{\sin\alpha}{5+2k^2+k^4}y\right\}\left(x^2+\frac{y^2}{k^2}-1\right)^2 \tag{45}$$

Using (33), i.e., $\omega_0 = \frac{c}{4}\left[z\bar{z} - D\left(z^2+z^{-2}\right) - E\right]$ and (37) in the equation (26), we get

$$\frac{\partial^2\omega_1}{\partial z\partial\bar{z}} = \frac{-ic}{8}\begin{bmatrix} \left(AD^2+2BD^3\right)z^5 - \left(BD^2+2AD^3\right)z^{-5} - \left(2AD+5Bd^2+2AD^3\right)z^4\bar{z} \\ +\left(2BD+5AD^2+2BD^3\right)z\bar{z}^4 + \left(A+4BD+6AD^2+4BD^3\right)z^3\bar{z}^{-2} \\ -\left(B+4AD+6BD^2+4AD^3\right)z^2\bar{z}^{-3} - \left(2AD+2BD^2\right)Ez^3 \\ -2\left(BD+2AD^2\right)E\bar{z}^3 - 2\left(BD+2AD^2\right)E\bar{z}^3 - \left(2A+6BD+4AD^2\right)Ez^2\bar{z} \\ +\left(2B+6AD+4BD^2\right)Ez\bar{z}^2 + \left(A+2BD\right)E^2z - \left(B+2AD\right)E^2\bar{z} \end{bmatrix} \tag{46}$$

Let the solution of the equation (46) be assumed in the form

$$\omega_1 = \begin{pmatrix} A_0z^5 + A_1z^4\bar{Z} + A_2z^3\bar{Z}^2 + A_3z^2\bar{Z}^3 + A_4z\bar{Z}^{-4} \\ +A_5z^{-5} + A_6z^4 + A_7z^3\bar{Z} + A_8z^2\bar{Z}^2 + A_9z\bar{Z}^3 \\ +A_{10}\bar{Z}^{-4} + A_{11}z^3 + A_{12}z^2\bar{Z} + A_{13}z\bar{Z}^2 + A_{14}\bar{Z}^3 \\ +A_{15}z^2 + A_{16}z\bar{Z} + A_{17}\bar{Z}^2 + A_{18}z + A_{19}\bar{Z} + A_{20} \end{pmatrix}\left\{z\bar{Z} - D\left(z^2+\bar{Z}^2\right) - E\right\}, \tag{47}$$

where A are constant to be determined such that (47) satisfy the equation (46). Then

$$\frac{\partial^2\omega_1}{\partial z\partial\bar{z}} = \begin{bmatrix} 6\left(A_0-DA_1\right)z^5 + 6\left(A_5-DA_4\right)z^{-5} + 10\left(A_1-DA_2-DA_0\right)z^4\bar{Z} \\ +10\left(A_4-DA_3-DA_5\right)\bar{z}^{-4}z + 12\left(A_2-DA_1-DA_3\right)z^3\bar{z}^2 \\ 12\left(A_3-DA_2-DA_4\right)z^2\bar{z}^3 + 8\left(A_7-DA_6-DA_8\right)z^3\bar{Z} \\ +8\left(A_9-DA_8-DA_{10}\right)z\bar{z}^3 + 9\left(A_8-DA_7-DA_9\right)z^2\bar{z}^2 \\ +4\left(A_{11}-DA_{12}-EA_1\right)z^3 + 6\left(A_{12}-DA_{11}-DA_{13}-EA_2\right)z^2\bar{Z} \\ +5\left(A_6-DA_7\right)z^4 + 10\left(A_{10}-DA_9\right)\bar{z}^3 + 4\left(A_{14}-DA_{13}-EA_4\right)\bar{z}^3 \\ +6\left(A_{13}-DA_{12}-DA_{14}-EA_3\right)z\bar{z}^3 + 3\left(A_{15}-DA_{16}-EA_7\right)z^2 \\ +3\left(A_{17}-DA_{16}-EA_9\right)\bar{z}^2 + 4\left(A_{16}-DA_{15}-DA_{17}-EA_8\right)z\bar{z} \\ +2\left(A_{18}-DA_{19}-EA_{12}\right)z + 2\left(A_{19}-DA_{18}-EA_{13}\right)\bar{z} \\ +4\left(A_{20}-EA_{16}\right) \end{bmatrix} \tag{48}$$

Comparing like coefficient in (46) and (48) we obtain a set of (44) equation for the As. Solving these by matrix inverse method, we get

$$A_0 = \frac{-ic^2k^2}{92160\Delta_1}\begin{bmatrix} \Big\{18\left(k^{18}-1\right)+414k^2\left(k^{14}-1\right)-1096k^4\left(k^{10}-1\right) \\ -1576k^6\left(k^6-1\right)-3812k^8\left(k^2-1\right)\Big\}e^{i\alpha} \\ -\Big\{32\left(k^{18}+1\right)+736k^2\left(k^{14}+1\right)+2496k^4\left(k^{10}+1\right) \\ +64k^6\left(k^6+1\right)-3328k^8\left(k^2+1\right)\Big\}e^{-i\alpha} \end{bmatrix}$$

$$A_1 = \frac{-ic^2k^2}{92160\Delta_1}\begin{bmatrix} \{104\left(k^{18}+1\right)+2600k^2\left(k^{14}+1\right)+12320k^4\left(k^{10}+1\right) \\ +7840k^6\left(k^6-1\right)-2286k^8\left(k^2+1\right)\}e^{i\alpha} \\ -\{146\left(k^{18}-1\right)+3790k^2\left(k^{14}-1\right)+20216k^4\left(k^{10}-1\right) \\ +29896k^6\left(k^6-1\right)-10764k^8\left(k^2-1\right)\}e^{-i\alpha} \end{bmatrix}$$

$$A_2 = \frac{-ic^2k^2}{46080\Delta_1}\begin{bmatrix} \{118\left(k^{18}-1\right)+3138k^2\left(k^{14}-1\right)+18432k^4\left(k^{10}-1\right) \\ +36496k^6\left(k^6-1\right)+21084k^8\left(k^2-1\right)\}e^{i\alpha} \\ -\{132\left(k^{18}+1\right)+3572k^2\left(k^{14}+1\right)+22912k^4\left(k^{10}+1\right) \\ +58016k^6\left(k^6+1\right)+79208k^8\left(k^2+1\right)\}e^{-i\alpha} \end{bmatrix}$$

$$A_3 = -\bar{A}_2; A_4 = -\bar{A}_1; A_5 = -\bar{A}_0; A_6 = A_7 = A_8 = A_9 = A_{10} = 0$$

$$A_{11} = \frac{-ic^2k^2}{5760\Delta_2}\begin{bmatrix} \{-65\left(k^{24}+1\right)-2480k^2\left(k^{20}+1\right)-28618k^4\left(k^{16}+1\right)-114036k^6 \\ \left(k^{12}+1\right)-129171k^8\left(k^8+1\right)+116516k^{10}\left(k+1\right)+315708k^2\}e^{i\alpha} \\ -\{135\left(k^{24}-1\right)+4650k^2\left(k^{20}-1\right)+50612k^4\left(k^{16}-1\right) \\ +221202k^6\left(k^{12}-1\right)+422465k^8\left(k^8+1\right)+353148k^{10}\left(k^4-1\right)\}e^{-i\alpha} \end{bmatrix}$$

$$A_{12} = \frac{-ic^2k^2}{5760\Delta_2}\begin{bmatrix} \{265\left(k^{24}-1\right)+9790k^2\left(k^{20}-1\right)+121548k^4\left(k^{16}-1\right) \\ +665130k^6\left(k^{12}-1\right)+171886k^8\left(k^8-1\right)-1894476k^{10}\left(k^4-1\right)\}e^{i\alpha} \\ -\{335\left(k^{24}+1\right)+13500k^2\left(k^{20}+1\right)+18470k^4\left(k^{16}+1\right)+1130444k^6 \\ \left(k^{12}+1\right)+3599681k^8\left(k^8+1\right)+6725496k^{10}\left(k^4+1\right)+8148964k^{12}\}e^{-i\alpha} \end{bmatrix}$$

$$A_{13} = -\bar{A}_{12}; A_{14} = -\bar{A}_{11}; A_{15} = A_{16} = A_{17} = 0$$

$$A_{12} = \frac{-ic^2k^6}{5760\Delta_3}\begin{bmatrix} \{270\left(k^{26}-1\right)+1737k^2\left(k^{22}-1\right)+332564k^4\left(k^{18}-1\right) \\ +256980k^6\left(k^{14}-1\right)+9394250k^8\left(k^{10}-1\right)+16195134k^{10}\left(k^6-1\right) \\ +9096184k^{12}\left(k^2-1\right)\}e^{i\alpha}-\{2580\left(k^{26}+1\right)+99620k^2\left(k^{22}+1\right) \\ +1369656k^4\left(k^{18}+1\right)+9225720k^6\left(k^{14}+1\right)+35106684k^8\left(k^{10}+1\right) \\ +82030284k^{10}\left(k^6+1\right)+123823696k^{12}\left(k^2+1\right)\}e^{-i\alpha} \end{bmatrix}$$

$A_{19} = -A_{18}; A_{20} = 0$

In above $\bar{A}$'s are conjugates of A.

$$\Delta_1 = \left(1+k^2\right)\left(5+2k^2+k^4\right)\left(5k^4+2k^2+1\right)\left(1+21k^2+35k^4+7k^6\right)\left(k^6+21k^4+35k^2\right)$$

$$\Delta_2 = \left(1+10k^2+5k^2\right)\left(k^4+10k^2+5\right)\Delta_1$$

$$\Delta_3 = \left(k^2+3\right)\left(1+3k^2\right)\Delta_2$$

Substituting these values in (47) we obtain the expression for ω_1 as

$$\omega_1 = \begin{bmatrix} A_0 z^5 + A_1 z^4 \bar{z} + A_2 z^3 \bar{z}^2 - \bar{A}_2 z^2 \bar{z}^3 - \bar{A}_1 z \bar{z}^4 - \bar{A}_0 \bar{z}^5 \\ + A_{11} z^3 + A_{12} z^2 \bar{z} - \bar{A}_{12} z \bar{z}^2 - \bar{A}_{11} \bar{z}^3 + A_{18} z + \bar{A}_{18} \bar{z} \end{bmatrix} \left[z\bar{z} - D\left(z^2 + z^{-2}\right) - E\right] \quad (49)$$

which converts into Cartesian form as

$$\omega_1 = \frac{c^2 k^6 \sin\alpha}{24(1+k^2)^2\left(5+2k^2+k^4\right)} x \left(x^2 + \frac{y^2}{k^2} - 1\right)$$
$$\left(\alpha_1 x^4 + \alpha_2 x^2 y^2 + \alpha_3 y^4 + \alpha_4 x^2 + \alpha_5 y^2 + \alpha_6\right) - \frac{c^2 k^6 \cos\alpha}{24(1+k^2)^2\left(1+2k^2+5k^4\right)} y$$
$$\left(x^2 + \frac{y^2}{k^2} - 1\right)\left(\beta_1 x^4 + \beta_2 x^2 y^2 + \beta_3 y^4 + \beta_4 x^2 + \beta_5 y^2 + \beta_6\right)$$

The constant $\alpha_1, \alpha_2, \cdots, \alpha_6$ and $\beta_1, \beta_2, \cdots, \beta_6$ in the above expression are

$$\alpha_1 = \frac{11k^2 + 24k^4 + 5k^6}{30\left(1+21k^2+35k^4+7k^6\right)}$$

$$\alpha_2 = \frac{4 + 60k^2 + 16k^4}{30\left(1+21k^2+35k^4+7k^6\right)}$$

$$\alpha_3 = \frac{1 + 20k^2 + 19k^4}{30\left(1+21k^2+35k^4+7k^6\right)k^2}$$

$$\alpha_4 = \frac{-2\left(11k^2 + 195k^4 + 449k^6 + 265k^8 + 40k^{10}\right)}{30\left(1+10k^2+5k^4\right)\left(1+21k^2+35k^4+7k^6\right)}$$

$$\alpha_5 = \frac{-2\left(2 + 55k^2 + 363k^4 + 445k^6 + 95k^8\right)}{30\left(1+10k^2+5k^4\right)\left(1+21k^2+35k^4+7k^6\right)}$$

$$\alpha_6 = \frac{k^2\left(11 + 289k^2 + 1854k^4 + 3346k^6 + 1895k^8 + 285k^{10}\right)}{30\left(1+3k^2\right)\left(1+10k^2+5k^4\right)\left(1+21k^2+35k^4+7k^6\right)}$$

$$\beta_1 = \frac{k^6 + 24k^4 + 19k^2}{30\left(k^6+21k^4+35k^2+7\right)}$$

$$\beta_2 = \frac{4\left(k^4 + 15k^2 + 4\right)}{30\left(k^6+21k^4+35k^2+7\right)}$$

$$\beta_3 = \frac{(11k^4 + 24k^2 + 5)}{30k^2 (k^6 + 21k^4 + 35k^2 + 7)}$$
$$\beta_4 = \frac{-2 (2k^{10} + 55k^8 + 363k^6 + 445k^4 + 95k^2)}{30 (k^4 + 10k^2 + 5) (k^6 + 21k^4 + 35k^2 + 7)}$$
$$\beta_5 = \frac{-2 (11k^8 + 195k^6 + 449k^4 + 265k^2 + 40)}{30 (k^4 + 10k^2 + 5) (k^6 + 21k^4 + 35k^2 + 7)}$$
$$\beta_6 = \frac{k^2 (11k^{10} + 289k^8 + 1854k^6 + 3346k^4 + 1895k^2 + 285)}{30 (k^2 + 3) (k^4 + 10k^2 + 5) (k^6 + 21k^4 + 35k^2 + 7)}$$

and coincides with that of [4]. The secondary flow in the cross section due to the pipe is given by the function ψ_1 in the equation (45). The corresponding velocity components are

$$\begin{aligned} u_1 &= -i \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}} \right) \bar{\psi}_1 \\ &= \frac{-i\omega_0}{\Delta_4} \left[\begin{array}{c} \{(9k^6 - 15k^4 + 11k^2 - 5)\, z^2 + (k^6 - 31k^4 - 29k^2 - 5)\, \bar{z}^2 \\ -2 (5k^6 + 27k^4 + 11k^2 + 5)\, z\bar{Z} - 4k^2 (5k^4 + 2k^2 + 1)\}\, e^{i\alpha} \\ \{(k^6 - 31k^4 - 29k^2 - 5)\, z^2 + (9k^6 - 15k^4 + 11k^2 - 5)\, \bar{z}^2 \\ -2 (5k^6 + 27k^4 + 11k^2 + 5)\, z\bar{z} - 4k^2 (5k^4 + 2k^2 + 1)\}\, e^{-i\alpha} \end{array} \right] \\ &= \frac{8k^2\omega_0}{\Delta_4} \left\{ 4 \left(5+2k^2+k^4\right) xy \cos\alpha + \left(1+2k^2+5k^4\right) \left[x^2 - \frac{5y^2}{k^2} - 1 \right] \sin\alpha \right\} \end{aligned} \tag{50}$$

and

$$\begin{aligned} v_1 &= \left(\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} \right) \bar{\psi}_1 \\ &= \frac{\omega_0}{\Delta_4} \left[\begin{array}{c} \{(5k^8 - 11k^6 + 15k^4 + 9k^2)\, z^2 + (5k^8 + 29k^6 + 31k^4 - k^2)\, \bar{z}^2 \\ -2 (5k^8 + 9k^6 + 23k^4 - 5k^2)\, z\bar{Z} - 4k^4 (5 + 2k^2 + k^4)\}\, e^{i\alpha} \\ \{(5k^8 + 29k^6 + 31k^4 - k^2)\, z^2 + (5k^8 - 11k^6 + 15k^4 - 9k^2)\, \bar{z}^2 \\ -2 (5k^8 + 9k^6 + 23k^4 - 5k^2)\, z\bar{z} - 4k^4 (5 + 2k^2 + k^4)\}\, e^{-i\alpha} \end{array} \right] \\ &= \frac{8k^2\omega_0}{\Delta_4} \left\{ k^2 \left(5+2k^2+k^4\right) \left(5x^2 + \frac{y^2}{k^2} - 1 \right) \cos\alpha + 4 \left(1+2k^2+5k^4\right) xy \sin\alpha \right\} \end{aligned} \tag{51}$$

Where ω_0 is given by (33) in the Cartesian form by (34), and

$$\Delta_4 = 96 \left(1 + 2k^2 + 5k^4\right) \left(5 + 2k^2 + k^4\right) \tag{52}$$

4. Discussion on the result

From the function ψ_1, it is evident that the secondary flow divides itself into two and that the plane of the separation is given by

$$\frac{k^2\cos\alpha}{1+2k^2+5k^4}x+\frac{\sin\alpha}{5+2k^2+k^4}y=0 \tag{53}$$

When the rotation o the pipe is about the major axis (corresponding to $\alpha=0$) or the minor axis (corresponding to $\alpha=\frac{\pi}{2}$), the plane of separation is perpendicular to the rotation axis. However, in the general case ($\alpha\neq 0,\frac{\pi}{2}$) it depends on the direction angle α as well as the eccentricity of the cross-section. The stream lines of the flow are given by the differential equations

$$\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w} \tag{54}$$

and in the present case we have, therefore

$$\frac{dx}{u_1}=\frac{dy}{v_1}=\frac{R}{a}\frac{dz}{w_0} \tag{55}$$

Using the expressions for ω_0, u_1 and v_1 from the equation (45), (51) and (52) respectively in equation (55) we see that the in the cross-section ($z=0$) the stream lines are given by

$$\left\{\frac{k^2\cos\alpha}{1+2k^2+5k^4}x+\frac{\sin\alpha}{5+2k^2+k^4}y\right\}\left(x^2+\frac{y^2}{k^2}-1\right)^2=cons\tan t \tag{56}$$

A particular of the fluid on the separation plane given in equation (53) stick to it in subsequent motion and flows on either side of this plane are distinct from each other. On the separation plane, the stream lines are given by

$$\frac{R}{a}dz=\frac{\omega_0}{u_1}dx=\frac{\omega_0}{v_1}dy \tag{57}$$

After integrating this equation we find that

$$z-z_0=\frac{6am}{Rk^2\sin\alpha}\left(5+2k^2+k^4\right)\log\left(\frac{m-x}{m+x}\right) \tag{58}$$

with

$$m^2=\frac{\left(1+2k^2+5k^4\right)^2\sin^2\alpha}{\left(1+2k^2+5k^4\right)^2\sin^2\alpha+k^2\left(5+2k^2+k^4\right)^2\cos^2\alpha} \tag{59}$$

The constant of integration z is different for different stream lines and measures the distance (parallel to the pipe) of the point corresponding to $x = 0$ on the stream line from the axis of rotation. When $x \to \pm m, z$ becomes infinitely large and hence no stream line in the separation plane ever reaches the edge of the pipe. This is seen to be true from any direction of the axis of rotation in the range $0 \leq \alpha \leq \frac{\pi}{2}$. When the pipe rotates about the major axis, we have $\alpha = 0$ and the stream function for the secondary flow is given by

$$\psi_1 = \frac{-ck^6 x}{24\left(1+k^2\right)\left(1+2k^2+5k^4\right)}\left(x^2 + \frac{y^2}{k^2-1}\right)^2 \tag{60}$$

The separation plane is now given by $x = 0$ and on this plane; the stream line is given by

$$\frac{k^3 R}{6\left(1+2k^2+5k^4\right)a}\left(z - z_0\right) = \log\left(\frac{k+y}{k-y}\right) \tag{61}$$

When $y \to \pm k, z$ becomes infinitely large, when $\alpha = \frac{\pi}{2}$ the axis of rotation coincides with the minor axis and stream function of the secondary flow is

$$\psi_1 = \frac{-ck^4 y\left(\frac{x^2+y^2}{k^2-1}\right)^2}{24\left(1+k^2\right)\left(5+2k^2+k^4\right)} \tag{62}$$

The separation plane in this case is $y = 0$ and on this plane the stream lines are given by

$$z - z_0 = \frac{6a\left(5+2k^2+k^4\right)}{Rk^2}\log\left(\frac{1-x}{1+x}\right) \tag{63}$$

When $x \to \pm 1, z$ becomes infinitely large. The velocity components u_1, v_1 of the secondary flow are given above in equation (3.28) and (3.29). We see that on the diameter

$$y = mx = \frac{1+2k^2+5k^4}{\left(5+2k^2+k^4\right)}x\tan\alpha \tag{64}$$

there are two points, where $u_1, v_1 = 0$ and the motion at theses point is entirely along the length of the pipe. These points are given by

$$x^2 = x_0^2 = \frac{k^2\left(5+2k^2+k^4\right)}{5\left\{k^2(5+2k^2+k^4)^2 + \left(1+2k^2+5k^4\right)\tan^2\alpha\right\}} \tag{65}$$

The two points $(x_{0,}mx_0)$ and $(-x_{0,} - mx_0)$ which are thus stagnation point of the secondary flow, are found to be on the diameter of the cross-section of the pipe which is conjugate to the diameter along which the separation plane of the secondary flow interest the cross-section. It is further seen easily that each stagnation point divides the semi-diameter passing through it in the ratio $1 : \sqrt{5}$. When the axis of rotation is a principal axis, we have $\alpha = 0$ or $\frac{\pi}{2}$ and the stagnation points will occur on the axis of rotation itself. The figures1, 2, 3 show the stream lines of the secondary flow for the cases (i) $\alpha = 0, k = 0.1$(ii) $\alpha = \frac{\pi}{2}, k = 0.9$(iii) $\alpha = \frac{\pi}{4}, k = 0.5$ marked on one side of the separation plane (AB). The stagnation point is indicated by a cross in the figures. From equation (4:3) above we find that the stream lines of the flow are given by

$$\frac{R}{a} dz = \frac{dx}{\frac{u_1}{\omega_0}} = \frac{dy}{\frac{v_1}{\omega_0}} \tag{66}$$

To examine the nature of the stream lines near the edge of the pipe, we write the above equation in terms of the polar coordinates (r, θ) defined by means of the equation

$$x = e + r\cos\theta,\ y = r\sin\theta \tag{67}$$

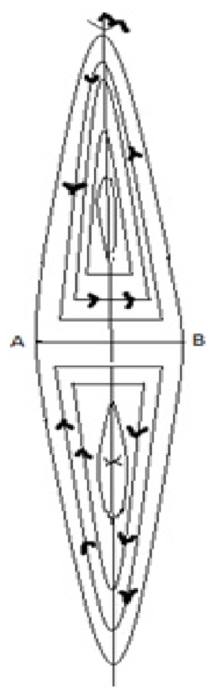

FIGURE 1. Secondary flow for $\alpha = 0, k = 0.1$

where $e = \sqrt{(1-k^2)}$ is the eccentricity of the ellipse. From equation (66), we find that

$$\frac{k^2R}{12a}dz = \frac{-rd\theta}{D} \tag{68}$$

where

$$D = \frac{cos\alpha}{(1+2k^2+5k^4)}\begin{bmatrix} 4\,(e+r\cos\theta)\,r\text{sin}^2\theta + 5k^2(e+r\cos\theta)^2\cos\theta \\ +r^2\text{sin}^2\theta\cos\theta - k^2\cos\theta \end{bmatrix} + \frac{\sin\alpha}{(5+2k^2+k^4)}\begin{bmatrix} (e+r\cos\theta)^2\sin\theta + \left(\frac{5}{k^2}\right)r^2\text{sin}^3\theta - \sin\theta \\ +4\,(e+r\cos\theta)\,r\sin\theta\cos\theta \end{bmatrix} \tag{69}$$

Near the edge of the pipe we may write

$$r \sim \frac{k^2}{1+e\cos\theta} \tag{70}$$

and from the equation (67), (68) we now have

$$\frac{k^2R}{3a}dx \sim dfrac{-}d\theta\frac{\cos\alpha\,(e+\cos\theta)}{1+2k^2+5k^4} + \frac{\sin\alpha\sin\theta}{5+2k^2+k^4} \tag{71}$$

Integrating this we get

$$\frac{k^2R}{3a}(z-z_1) \sim dfrac1\sqrt{(n^2+k^2)}\log\left\{\frac{\frac{\sqrt{n^2+k^2}}{m\,(1-e^2)} - \left(t - \frac{n}{m\,(1-e)}\right)}{\frac{\sqrt{n^2+k^2}}{m\,(1-e^2)} + \left(t - \frac{n}{m\,(1-e)}\right)}\right. \tag{72}$$

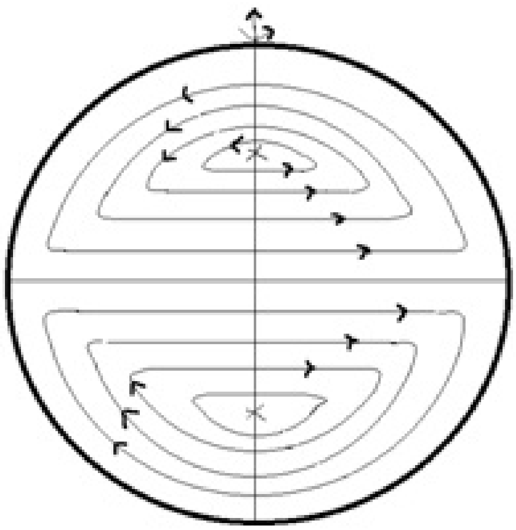

Figure 2. Secondary flow for $a = \frac{\pi}{2}, k = 0.9$

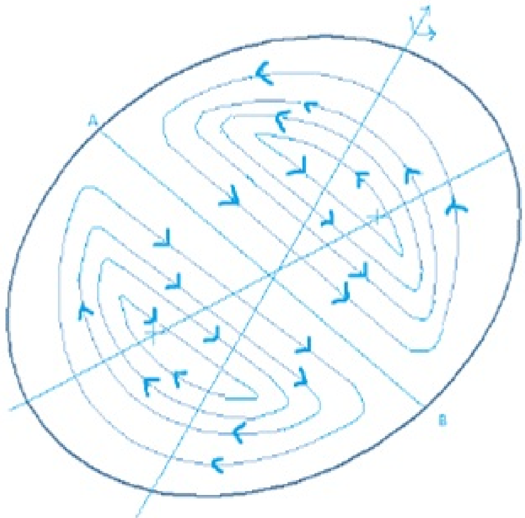

FIGURE 3. Secondary flow for $a = \frac{\pi}{4}, k = 0.5$

where
$$m = \frac{\cos\alpha}{1+2k^2+5k^4}, n = \frac{\sin\alpha}{5+2k^2+k^4}, t = \tan\frac{\theta}{2} \tag{73}$$
We see from (72) that the linear distance, which the fluid particle has to travel to reach from $t = t_0$ to $t = T$ is inversely proportional to the angular speed of the pipe. When the pipe rotates about either of the principal axes, the equation (72) gets modified. For $\alpha = 0$ (the rotation about the major axis), we have

$$z - z_1 \sim dfrac3\left(1+2k^2+5k^4\right) ak^3 R \log\left\{\frac{\sqrt{\frac{1+e}{1-e}} - \tan\frac{\theta}{2}}{\sqrt{\frac{1+e}{1-e}} + \tan\frac{\theta}{2}}\right\} \tag{74}$$

and when $\alpha = \frac{\pi}{2}$ (rotation about the minor axis), we have

$$z - z_1 \sim dfrac3\left(5+2k^2+k^4\right) ak^2 R \log\, cot\frac{\theta}{2} \tag{75}$$

Resistance coefficient The flux through the pipe is given by

$$F = -\iint W(\varepsilon,\eta)\, d\varepsilon\, d\eta = -va\iint w(x,y)\, dx\, dy \tag{76}$$

For a stationary elliptic pipe, we have the flux given by

$$F_s = \frac{\pi k^3 cva}{4\left(1+k^2\right)} = \pi a^2 k w_m \tag{77}$$

on using the expression for w from the equation (45) in (76) above. In the equation (77) w_m denotes the mean velocity through the stationary pipe, corresponding to the constant pressure gradient c .

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