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# On the Oscillation of Non-linear Functional Partial Differential Equations

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Abstract. In this article, we investigate the oscillatory behavior of nonlinear partial differential equations (1) with the boundary condition (2). By using integral averaging method, we will obtain some new oscillation criteria for given system. The main results are illustrated through suitable example.

**Key words:** Oscillation, Partial differential equations, Delay differential equations.

**AMS Subject classification:** 34L30, 35B05.

## 1. Introduction

Differential equations have a remarkable ability to predict the world around us. Partial Differential equations form an essential part of the core Mathematics for scientists and engineering. The origins and applications of such equations occur in a variety of different fields, such as fluid dynamics, heat conduction and diffusion, to describe the motion of waves in physics, modeling chemical reactions in chemistry, the population growth of species. We refer the monographs in the literature [1, 5, 8, 12, 15, 17]. The qualitative theory of partial differential equations has attracted a great deal of attention over the last few decades. See for example [9–11, 13, 14, 16] and the references cited therein.

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In [12], the study of entry-flow phenomenon as a problem of hydrodynamics is differential equations of the following form

$$x^{'''} + a(t)x^{''} + b(t)x^{'} + c(t)x = f(t)$$

occurs in many branches of engineering. In the last two decades, there has been a lot of attention shown on several aspects of differential equations of third order [2, 6, 7].

Agarwal et al. [3] and Aktas et al. [4] investigated the oscillatory behavior of nonlinear delay differential equations of the form

$$\Big(r_2(t)\big(r_1(t)x^{'}\big)^{'}\Big)^{'} + p(t)x^{'} + q(t)f(x(g(t))) = 0.$$

However, there has been no work done on nonlinear partial functional differential equations given in (1). This motivated our research work.

**Formulation of the problem:**

In the present article, we consider the oscillatory behavior of functional partial differential equations of the form

$$\frac{\partial}{\partial t}\Big(\frac{1}{r(t)}\frac{\partial}{\partial t}\Big(\frac{1}{p(t)}\Big(\frac{\partial}{\partial t}u(x,t)\Big)^{\gamma}\Big)\Big) + q(x,t)f\left(u(x,\tau(t))\right)$$
$$= a(t)\Delta u(x,t) + F(x,t),\ (x,t) \in \Omega \times \mathbb{R}_+ = G, \tag{1}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N$ with a piecewise smooth boundary $\partial\Omega$, $\gamma$ is the ratio of odd positive integers and $\Delta$ is the Laplacian operator in the Euclidean $N$- space $\mathbb{R}^N$, $\Delta u(x,t) = \sum_{r=1}^{N} \dfrac{\partial^2 u(x,t)}{\partial x_r^2}$ with the Robin boundary condition

$$\frac{\partial u(x,t)}{\partial \nu} + \mu(x,t)u(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}_+. \tag{2}$$

where $\nu$ is the unit exterior normal vector to $\partial\Omega$ and $\mu(x,t)$ is positive continuous function on $\partial\Omega \times \mathbb{R}_+$.

We shall assume throughout this paper that:

$(A_1)$ $r(t) \in C^1([0,\infty);[0,\infty))$, $r(t) > 0$, $p(t) \in C^2([0,\infty);[0,\infty))$, $p(t) > 0$, $a(t) \in C([0,\infty);[0,\infty))$ *and* $\int^{\infty} p^{\frac{1}{\gamma}}(s)ds = \infty$;

$(A_2)$ $q(x,t) \in C(\bar{G};[0,\infty))$, $Q(t) = min_{x\in\bar{\Omega}}q(x,t)$ *and* $sup\,\{q(t) : t \geq T\} > 0$ for any $T \geq t_0 \geq 0$;

$(A_3)$ $f \in C(\mathbb{R};\mathbb{R})$ are convex in $[0,\infty)$ with $uf(u) > 0,\ f'(u) \geq 0$ for $u \neq 0$;

$(A_4)$ $F \in C(\bar{G};\mathbb{R})$ such that $\int_\Omega F(x,t)dx \leq 0$;

$(A_5)$ $\tau \in C^1([0,\infty);\mathbb{R})$ satisfying $\tau'(t) \geq 0,\ \tau(t) < t$ and $\lim\limits_{t\to\infty} \tau(t) = \infty$.

**Definition:** A function $u \in C^2(G) \cap C^1(\overline{G})$ is called a solution of (1) and (2) if it satisfies (1) in G and the boundary condition (2). The solution u(x,t) of (1) and (2) is oscillatory in the domain G if for any positive number $\lambda$ there exists a point $(x_0, y_0) \in \Omega \times [\lambda, \infty)$ such that $u(x_0, y_0) = 0$ holds.

The main purpose of this paper is to establish some new oscillation criteria for (1) and (2) by using integral averaging method. Our results are essentially new.

## 2. Main Results

We use the following notations throughout this paper.

$$v(t) = \int_\Omega u(x,t)dx \quad \text{and } \Psi(t) = \int_{t_0}^t r(s)ds. \tag{3}$$

Now, we present some new oscillation results.

**Theorem 2.1.** *If the differential inequality*

$$\frac{d}{dt}\left(\frac{1}{r(t)}\frac{d}{dt}\left(\frac{1}{p(t)}\left(\frac{d}{dt}v(t)\right)^\gamma\right)\right) + Q(t)f(v[\tau(t)]) \leq 0,\ t \geq t_0 \tag{4}$$

*has no eventually positive solution, then every solution of equation* (1) *and* (2) *is oscillatory in* $\Omega \times \mathbb{R}_+$.

*Proof.* Assume for the sake of contradiction that there is a nonoscillatory solution $u(x,t)$ of (1) and (2) which has no zero in $\Omega \times [0,\infty)$ for some $t_0 > 0$. Then $u(x,t) > 0$ for $t \geq t_0$. Integrating (1) with respect to $x$ over $\Omega$, we have

$$\int_\Omega \left(\frac{d}{dt}\left(\frac{1}{r(t)}\frac{d}{dt}\left(\frac{1}{p(t)}\left(\frac{d}{dt}u(x,t)\right)^\gamma\right)\right)\right)dx + \int_\Omega q(x,t)f\left(u(x,\tau(t)\right)dx$$

$$= \int_\Omega a(t)\Delta u(x,t)dx + \int_\Omega F(x,t)dx. \tag{5}$$

By Jensen's inequality we get

$$\int_\Omega \left(\frac{d}{dt}\left(\frac{1}{r(t)}\frac{d}{dt}\left(\frac{1}{p(t)}\left(\frac{d}{dt}u(x,t)\right)^\gamma\right)\right)\right)dx$$
$$\geq \frac{d}{dt}\left(\frac{1}{r(t)}\frac{d}{dt}\left(\frac{1}{p(t)}\int_\Omega\left(\frac{d}{dt}u(x,t)\right)^\gamma dx\right)\right)$$
$$\geq \frac{d}{dt}\left(\frac{1}{r(t)}\frac{d}{dt}\left(\frac{1}{p(t)}\left(\frac{d}{dt}v(t)\right)^\gamma\right)\right),\ t \geq t_0, \tag{6}$$

again Jensen's inequality and $(A_2)$ gives,

$$\int_\Omega q(x,t)f\left(u(x,\tau(t)\right)dx \geq Q(t)\int_\Omega f\left(u(x,\tau(t)\right)dx \geq Q(t)f(v[\tau(t)]), \tag{7}$$

also using Green's formula and (2), we get

$$\int_\Omega \Delta u(x,t)dx = \int_{\partial\Omega}\frac{\partial u(x,t)}{\partial \nu}dS = -\int_{\partial\Omega}\mu(x,t)u(x,t)dS \leq 0, \quad t \geq t_0, \tag{8}$$

In view of $(3),(6)-(8),(A_4)$ and (5) yield

$$\frac{d}{dt}\left(\frac{1}{r(t)}\frac{d}{dt}\left(\frac{1}{p(t)}\left(\frac{d}{dt}v(t)\right)^\gamma\right)\right) + Q(t)f(v[\tau(t)]) \leq 0, \quad t \geq t_0. \qquad \square$$

**To construct the operators to the inequality** (4)

Define the operators

$$L_0v(t) = v(t), \qquad L_1v(t) = \frac{1}{p(t)}\left(\frac{d}{dt}L_0v(t)\right)^\gamma,$$
$$L_2v(t) = \frac{1}{r(t)}\frac{d}{dt}L_1v(t), \qquad L_3v(t) = \frac{d}{dt}L_2v(t). \tag{9}$$

Thus inequality (4) becomes

$$L_3v(t) + Q(t)f\left(v[g(t)]\right) \leq 0.$$

Let us assume that there is a nonoscillatory $v(t)$ of (4). With out loss of generality, it is further assume that $v(t)$ be an eventually positive solution of (4), then $L_3v(t) \leq 0$ eventually, and hence $L_iv(t),\ i = 0,1,2$ are eventually of one sign.

Here arise two possible cases:

$(I)$ $L_iv(t) > 0,\ i = 0,1,2$ are eventually, or

$(II)$ $L_0v(t) > 0,\ L_1v(t) < 0$ and $L_2v(t) > 0$ eventually.

Case(I) Let $L_i v(t) > 0,\ i = 0, 1, 2$ for $t \geq t_0 \geq 0$. Then, from (9) we obtain that

$$L_1 v(t) = \int_{t_0}^{t} L_2 v(s) r(s) ds \geq L_2 v(t) \int_{t_0}^{t} r(s) ds \geq L_2 v(t) \psi(t)\ for\ t \geq t_0,$$

or

$$v^{'}(t) \geq p^{\frac{1}{\gamma}}(t) \psi^{\frac{1}{\gamma}}(t) L_2^{\frac{1}{\gamma}} v(t),\ t \geq t_0.$$

Integrating from $t_0$ to t, we have

$$v(t) \geq L_2^{\frac{1}{\gamma}} v(t) \left( \int_{t_0}^{t} p^{\frac{1}{\gamma}}(s) \psi^{\frac{1}{\gamma}}(s) ds \right).$$

Let us take $D_1[t, t_0] = \int_{t_0}^{t} p^{\frac{1}{\gamma}}(s) \psi^{\frac{1}{\gamma}}(s) ds$, then

$$v(t) \geq D_1[t, t_0] L_2^{\frac{1}{\gamma}} v(t)\ for\ t \geq t_0. \tag{10}$$

Case(II) Let $L_0 v(t) > 0,\ L_1 v(t) < 0$ and $L_2 v(t) > 0,\ t \geq t_0 \geq 0$. Then, for $t \geq s \geq t_0$, which yields that

$$L_1 v(t) - L_1 v(s) = \int_{s}^{t} L_2 v(u) r(u) du \geq L_2 v(t) \int_{s}^{t} r(u) du,$$

$$-L_1 v(s) \geq \psi(t) L_2 v(t),$$

or

$$-v^{'}(s) \geq p^{\frac{1}{\gamma}}(s) \psi^{\frac{1}{\gamma}}(t) L_2^{\frac{1}{\gamma}} v(t)$$

Thus, we have

$$v(s) \geq L_2^{\frac{1}{\gamma}} v(t) \left( \int_{s}^{t} p^{\frac{1}{\gamma}}(\tau) \psi^{\frac{1}{\gamma}}(\tau) d\tau \right)$$

Let $D_2[t, s] = \int_{s}^{t} p^{\frac{1}{\gamma}}(\tau) \psi^{\frac{1}{\gamma}}(\tau) d\tau$, then

$$v(s) \geq L_2^{\frac{1}{\gamma}} v(t) D_2[t, s]\ for\ t \geq s \geq t_0, \tag{11}$$

Also assume that

$$-\phi(-xy) \geq \phi(xy) \geq \phi(x)\phi(y)\ for\ xy > 0, \tag{12}$$

$$\frac{\phi(u^{\frac{1}{\gamma}})}{u} \geq m > 0,\ m\ is\ a\ real\ constant,\ u \neq 0, \tag{13}$$

and

$$\int_{0}^{\pm\epsilon} \frac{du}{\phi(u^{\frac{1}{\gamma}})} < \infty\ forevery\ \epsilon > 0. \tag{14}$$

**Theorem 2.2.** *Assume that* $(A_1)$ *to* $(A_5)$, (12) *and* (13) *hold. If for* $t \geq t_0 \geq 0$,

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} Q(s) f(D_1[\tau(s), t_0]) ds > \frac{1}{m} \tag{15}$$

*and*

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} Q(s) f(D_2[\tau(t), \tau(s)]) ds > \frac{1}{m}, \tag{16}$$

*then all the solutions of* (1), (2) *is oscillatory in* $G$.

*Proof.* Let v(t) be an eventually positive solution of (4). Then, $L_3 v(t) \leq 0$ and in view of that $L_i v(t)$, $i = 1, 2, 3$ are eventually of one sign. Which gives two possibilities (I) and (II). For Case (I), we get (10). Now, there is a $T \geq t_0$ such that

$$v[\tau(t)] \geq D_1[\tau(t), t_0] L_2^{\frac{1}{\gamma}} v[\tau(t)] \ for \ t \geq T. \tag{17}$$

An integration for (4) from $\tau(t)$ to $t (\geq T)$ and from (17), we have

$$L_2 v(t) - L_2 v[\tau(t)] \leq -\int_{\tau(t)}^{t} Q(s) f(v[\tau(s)]) ds.$$

This implies that

$$L_2 v[\tau(t)] \geq f(L_2^{\frac{1}{\gamma}} v[\tau(t)]) \int_{\tau(t)}^{t} Q(s) f(D_1[\tau(s), t_0]) ds,$$

or

$$\frac{L_2 v[\tau(t)]}{f(L_2^{\frac{1}{\gamma}} v[\tau(t)])} \geq \int_{\tau(t)}^{t} Q(s) f(D_1[\tau(s), t_0]) ds.$$

Taking limsup on both sides as $t \to \infty$, we get a contradiction to (15).
Next, for case (II), replace $\tau(s)$ and $\tau(t)$ by s and t respectively in (11), we have

$$v[\tau(s)] \geq D_2[\tau(t), \tau(s)] L_2^{\frac{1}{\gamma}} v[\tau(t)] \ for \ t \geq s \geq t_0. \tag{18}$$

Integrating inequality (4) from $\tau(t)$ to t, the proof is same to Case (I), so the details are omitted. □

**Corollary 2.3.** *Suppose that the conditions* $(A_1) - (A_5)$, (12) *and* (13) *hold. If* (16) *holds, then all bounded solutions of* (1), (2) *is oscillatory in* $G$.

**Theorem 2.4.** *Assume that* $(A_1)$ *to* $(A_5)$, (12) *and* (14) *hold. If for* $t \geq t_0 \geq 0$,

$$\int^{\infty} Q(s)f(D_1[\tau(s), t_0])ds = \infty \tag{19}$$

*and*

$$\int^{\infty} Q(s)f(D_2[\tau(t), \tau(s)])ds = \infty, \tag{20}$$

*then all the solutions of* (1), (2) *is oscillatory in* $G$.

*Proof.* Let v(t) be an eventually positive solution of inequality (4). We can proceed as in the proof of Theorem 2.2. For Case (I), using (12) we have

$$\frac{-\frac{d}{dt}L_2 v(t)}{f(L_2^{\frac{1}{\gamma}} v(t))} \geq Q(t)f\left(D_1[\tau(t), t_0]\right) for\ t \geq T \geq t_0.$$

Integrating T to t, we have that

$$\int_{L_2 v(t)}^{L_2 v(T)} \frac{du}{f(u^{\frac{1}{\gamma}})} \geq \int_T^t Q(s)f(D_1[\tau(s), t_0])ds.$$

On both sides, taking limit as $t \to \infty$, we get a contradiction to (19).

Next, for Case (II), from (4), we have

$$-L_3 v(s) \geq Q(s)f(v[\tau(s)]) \geq Q(s)f(D_2[\tau(t), \tau(s)])f(L_2^{\frac{1}{\gamma}} v[\tau(s)])\ for\ t \geq s \geq T \geq t_0,$$

or

$$\frac{-\frac{d}{ds}L_2 v(s)}{f(L_2^{\frac{1}{\gamma}} v[\tau(s)])} \geq Q(s)f(D_2[\tau(t), \tau(s)]).$$

The proof is analogous to that of Case (I) and thus the details are omitted. $\square$

**Corollary 2.5.** *Assume that the conditions* $(A_1) - (A_5)$ *and* (12) *hold. If*

$$\frac{u}{f(u^{\frac{1}{\gamma}})} \to 0\ as\ u \to 0 \tag{21}$$

*and*

$$\limsup_{t \to \infty} \int_{\tau(t)}^{t} Q(s)f(D_2[\tau(t), \tau(s)])ds > 0, \tag{22}$$

*then all bounded solutions of* (1), (2) *is oscillatory in* $G$.

**Theorem 2.6.** *Suppose that the conditions* $(A_1)-(A_5)$ *and* (12) *hold. If the inequalities*

$$w^{'}(t)+Q(t)f\left(D_1[\tau(t),t_0]\right)f\left(w^{\frac{1}{\gamma}}[\tau(t)]\right)\le 0,\ t_0\ge 0 \tag{23}$$

*and*

$$y^{'}(t)+Q(t)f\left(D_2\left[\frac{t+\tau(t)}{2},\tau(t)\right]\right)f\left(y^{\frac{1}{\gamma}}\left[\frac{t+\tau(t)}{2}\right]\right)\le 0 \tag{24}$$

*are oscillatory, then all the solutions of* (1), (2) *is oscillatory.*

*Proof.* Let v(t) be an eventually positive solution of inequality (4). We can proceed as in the proof of Theorem 2.2. For Case (I),

$$\frac{d}{dt}L_2v(t)\le -Q(t)f(D_1[\tau(t),t_0])f(L_2^{\frac{1}{\gamma}}v[\tau(t)])\ for\ t\ge T\ge t_0.$$

Take $w(t)=L_2v(t)>0$ for $t\ge T$, we get (23).
Integrating (23) from t to u as u $\to\infty$, we obtain

$$w(t)\ge\int_t^{\infty}Q(s)f(D_1[\tau(s),t_0])f(w^{\frac{1}{\gamma}}[\tau(s)])ds,\ for\ t\ge T.$$

With this to conclude that there is a positive solution w(t) of (23) with $\lim_{t\to\infty}w(t)=0$, which is a contradiction to (23) and hence $v(t)$ is oscillatory.
Next, for Case (II), we get (11). Replacing $\tau(t)$ for s and $\frac{t+\tau(t)}{2}$ for t, we have

$$v[\tau(t)]\ge D_2\left[\frac{t+\tau(t)}{2},\tau(t)\right]y^{\frac{1}{\gamma}}\left[\frac{t+\tau(t)}{2}\right].$$

Using the above inequality in (4), take $y(t)=L_2v(t)>0$ and similar as in Case (I) above, we get (24). The remaining proof is related to Case (I) above and hence the details are omitted. □

**Corollary 2.7.** *Suppose that the conditions* $(A_1)-(A_5)$, (12) *and* (13) *hold. If*

$$\liminf_{t\to\infty}\int_{\tau(t)}^{t}Q(s)f(D_1[\tau(s),t_0])ds>\frac{1}{em},\ t_0\ge 0 \tag{25}$$

*and*

$$\liminf_{t\to\infty}\int_{\frac{t+\tau(t)}{2}}^{t}Q(s)f\left(D_2\left[\frac{t+\tau(t)}{2},\tau(t)\right]\right)ds>\frac{1}{em}, \tag{26}$$

*then all solutions of* (1), (2) *is oscillatory in G.*

## 3. Example

**Example 3.1.** *Consider the nonlinear partial delay differential equation*

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}u(x,t)\right)^3\right)+3\left(u(x,t-\frac{\pi}{2})\right)^3=\Delta u(x,t)-6e^{-3x}\cos^2 t\sin t$$

$$-e^{-x}\cos t \ for \ (x,t)\in(0,\pi)\times(0,\infty), \tag{27}$$

*with*

$$u_x(0,t)+u(0,t)=u_x(\pi,t)+u(\pi,t)=0, \ t\geq 0. \tag{28}$$

*Here* $\gamma=3,\ p(t)=r(t)=a(t)=1,\ q(x,t)=3,\ \tau(t)=t-\frac{\pi}{2},\ f(u)=u^\gamma$ *and* $F(x,t)=-6e^{-3x}\cos^2 t\sin t-e^{-x}\cos t$. *Also* $D_2[\tau(t),\tau(s)]=\frac{3}{4}(t-s)^{\frac{4}{3}}$,

$$\limsup_{t\to\infty}\int_{\tau(t)}^{t}3f(D_2[\tau(t),\tau(s)])ds=2.4206635>1.$$

*All the conditions of Theorem 2.2 are satisfied. Thus, every solution of (27), (28) is oscillatory in* $(0,\pi)\times(0,\infty)$. *In fact,* $u(x,t)=e^{-x}\cos t$ *is one such a solution.*

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