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On The Oscillation of Impulsive Neutral Partial Differential Equations with Distributed Deviating Arguments and Damping Term

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Abstract. In this work, we consider the oscillation of solutions of nonlinear impulsive neutral partial differential equations with distributed deviating arguments and damping term. Sufficient conditions are obtained for the oscillation of solutions using impulsive differential inequalities and integral averaging method with two boundary conditions. Example is given to illustrate our main results.

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1. Introduction

Oscillation theory of differential equations originated by C. Sturm [18] in 1836, and for partial differential equations by P. Hartman and A. Wintner [6] in 1955. Pioneer work on oscillation of impulsive delay differential equations [5] was published in 1989 and its results were included in monograph [7]. Likewise in 1991, the first work done in [4] on impulsive partial differential equations.

In [1] authors studied the asymptotic behavior of the nonoscillatory solutions of the neutral equations with distributed deviating arguments. Oscillatory properties of the nonlinear inhomogeneous hyperbolic equation with distributed deviating arguments investigated in [8]. Oscillatory properties of solutions of many partial differential equations with continuous distributed deviating arguments concentrated in [2, 3, 11, 13, 17, 19, 22, 23, 24] and monographs [25, 27]. Motivated by [9, 10] to introducing distributed deviating arguments for impulsive neutral parabolic partial differential equations. Particularly no work has been

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known with (E) and (B1) [(E) and (B2)] upto now. This paper generalizes many results of hyperbolic partial differential equations without impulse and distributed deviating arguments. Many authors studied the oscillation of partial differential equations with or wihtout impulse, see [15, 16, 14, 12, 26, 20] and the references cited therein. While comparing the importance between impulsive differential equations and corresponding differential equations, impulsive type has wide applications in various fields of science and technology.

In this paper, we focus our attention on oscillation of nonlinear impulsive neutral partial differential equations with distributed deviating arguments and damping term

$$\left.\begin{aligned} &\frac{\partial}{\partial t}\left[r(t)\frac{\partial}{\partial t}\left(u(x,t)+c(t)u(x,\tau(t))\right)\right]+p(t)\frac{\partial}{\partial t}\left(u(x,t)+c(t)u(x,\tau(t))\right)\\ &\quad+\int_a^b q(x,t,\xi)f(u(x,g(t,\xi)))d\eta(\xi)=a(t)\Delta u(x,t)\\ &\quad\quad-\int_a^b b(t,\xi)\Delta u(x,h(t,\xi))d\eta(\xi), \quad t\neq t_k, \quad (x,t)\in\Omega\times\mathbb{R}^+\equiv G\\ &u(x,t_k^+)=\ \alpha_k\left(x,t_k,u(x,t_k)\right),\\ &u_t(x,t_k^+)=\ \beta_k\left(x,t_k,u_t(x,t_k)\right), \quad t=t_k, \quad k=1,2,\cdots, \end{aligned}\right\} \quad (E)$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a piecewise smooth boundary $\partial\Omega$ and Δ is the Laplacian in the Euclidean space $\mathbb{R}^N$.

Equation (E) is supplemented by one of the following Dirichlet and Robin boundary conditions,

$$u=\ 0, \qquad (x,t)\in\partial\Omega\times\mathbb{R}^+ \qquad (B1)$$

$$\frac{\partial u}{\partial\gamma}+\mu(x,t)u=\ 0, \qquad (x,t)\in\partial\Omega\times\mathbb{R}^+ \qquad (B2)$$

where γ is the unit exterior normal vector to $\partial\Omega$ and $\mu(x,t)\in C(\partial\Omega\times\mathbb{R}^+,\ \mathbb{R}^+)$.

We assume that the following hypotheses (H) hold:

(H_1) $r(t)\in C'(\mathbb{R}^+,(0,+\infty))$, $r'(t)\geq 0$, $p(t)\in C(\mathbb{R}^+,\mathbb{R})$, $\int_{t_0}^{\infty}\frac{1}{R(s)}ds=\infty$, where $R(t)=\exp\left(\int_{t_0}^{t}\frac{r'(s)+p(s)}{r(s)}ds\right)$, $c(t)\in C^2(\mathbb{R}^+,\mathbb{R}^+)$, $a(t)\in PC(\mathbb{R}^+,\mathbb{R}^+)$, where PC denotes the class of functions which are piecewise continuous in

t with discontinuities of first kind only at $t = t_k,\ k = 1, 2, \cdots$, and left continuous at $t = t_k,\ k = 1, 2, \cdots$.

(H_2) $\tau(t) \in C(\mathbb{R}^+, \mathbb{R}),\ \lim_{t\to+\infty} \tau(t) = +\infty,\ q(x, t, \xi) \in C(\bar{\Omega} \times \mathbb{R}^+ \times [a, b], \mathbb{R}^+)$, $Q(t, \xi) = \min_{x\in\bar{\Omega}} q(x, t, \xi),\ f(u) \in C(\mathbb{R}, \mathbb{R})$ is convex in $\mathbb{R}^+$, $uf(u) > 0$ and $\frac{f(u)}{u} \geq \epsilon > 0$ for $u \neq 0$.

(H_3) $b(t, \xi) \in C(\mathbb{R}^+ \times [a, b], \mathbb{R}^+),\ \ g(t, \xi),\ \ h(t, \xi) \in C(\mathbb{R}^+ \times [a, b], \mathbb{R}),\ g(t, \xi)$, $h(t, \xi) \leq t$ for $\xi \in [a, b]$, $g(t, \xi)$ and $h(t, \xi)$ are nondecreasing with respect to t and ξ respectively and $\liminf_{t\to+\infty,\ \xi\in[a,b]} g(t, \xi) = \liminf_{t\to+\infty,\ \xi\in[a,b]} h(t, \xi) = +\infty$.

(H_4) There exist a function $\theta(t) \in C(\mathbb{R}^+, \mathbb{R}^+)$ satisfying $\theta(t) \leq g(t, a)$, $\theta'(t) > 0$ and $\lim_{t\to+\infty} \theta(t) = +\infty$, $\eta(\xi) \in ([a, b], \mathbb{R})$ is nondecreasing and the integral of equation (E) is a Stieltjes one.

(H_5) $u(x, t)$ and their derivative $u_t(x, t)$ are piecewise continuous in t with discontinuities of first kind only at $t = t_k$, $k = 1, 2, \cdots$, and left continuous at $t = t_k$, $u(x, t_k) = u(x, t_k^-)$, $u_t(x, t_k) = u_t(x, t_k^-)$, $k = 1, 2, \cdots$.

(H_6) $\alpha_k(x, t_k, u(x, t_k)),\ \beta_k(x, t_k, u_t(x, t_k)) \in PC(\bar{\Omega} \times \mathbb{R}^+ \times \mathbb{R}, \mathbb{R}),\ k = 1, 2, \cdots$, and there exist positive constants a_k, a_k^*, b_k, b_k^* with $b_k \leq a_k^*$ such that for $k = 1, 2, \cdots$

$$a_k^* \leq \frac{\alpha_k(x, t_k, u(x, t_k))}{u(x, t_k)} \leq a_k,$$

$$b_k^* \leq \frac{\beta_k(x, t_k, u_t(x, t_k))}{u_t(x, t_k)} \leq b_k.$$

This work is planned as follows: Section 2, we will give the definitions and notations. In Section 3, we deal with the oscillation of the problem (E) and $(B1)$. In Section 4, we discuss the oscillation of the problem (E) and $(B2)$. Section 5, presents an example to illustrate the main results.

2. Preliminaries

In this section, we introduce definitions and some well-known results which are needed throughout this paper.

Definition 2.1. A solution u of (E) is a function $u \in C^2(\bar{\Omega} \times [t_{-1}, +\infty), \mathbb{R}) \cap C(\bar{\Omega} \times [\tilde{t}_{-1}, +\infty), \mathbb{R})$ that satisfies (E), where

$$t_{-1} = \min\left\{0, \inf_{t\geq 0} \tau(t)\right\} \qquad \text{and}$$

$$\tilde{t}_{-1} = \min\left\{0, \min_{\xi\in[a,b]}\left\{\inf_{t\geq 0} g(t,\xi)\right\}, \min_{\xi\in[a,b]}\left\{\inf_{t\geq 0} h(t,\xi)\right\}\right\}.$$

Definition 2.2. For any function $k(t,s) \in C([t_0, +\infty) \times [t_0, t), \mathbb{R})$, $\sigma \geq t_0 \geq 0$, we define the linear integral operator $\mathcal{L}_\sigma^\rho$ as

$$\mathcal{L}_\sigma^\rho(k(t,s)) = \int_\sigma^t \rho(s)(t-s)^\alpha k(t,s)ds,$$

where $\alpha > 1$ is a constant, $\rho \in C'([t_0, +\infty), \mathbb{R})$ with $\rho > 0$.
If $\dfrac{\partial k(t,s)}{\partial s} \in C([t_0, +\infty) \times [t_0, t), \mathbb{R})$, we get

$$\mathcal{L}_\sigma^\rho\left(\frac{\partial k(t,s)}{\partial s}\right) = -\rho(\sigma)(t-\sigma)^\alpha k(t,\sigma) - \mathcal{L}_\sigma^\rho\left[\left(\frac{-\alpha}{t-s} + \frac{\rho'(s)}{\rho(s)}\right)k(t,s)\right].$$

Definition 2.3. The solution u of $(E), (B1)$ $[(E), (B2)]$ is said to be oscillatory in G if for any positive number ℓ there exist a point $(x_0, t_0) \in \Omega \times [\ell, +\infty)$ such that $u(x_0, t_0) = 0$ holds.

Definition 2.4. A function $V(t)$ is said to be eventually positive (negative) if there exists a $t_1 \geq t_0$ such that $V(t) > 0$ (< 0) holds for all $t \geq t_1$.

It is known that [21] the smallest eigenvalue $\lambda_0 > 0$ of the eigenvalue problem

$$\Delta\omega(x) + \lambda\omega(x) = 0 \qquad \text{in} \quad \Omega$$

$$\omega(x) = 0 \qquad \text{on} \quad \partial\Omega$$

and we can choose the corresponding eigenfunction $\Phi(x) > 0$ in Ω.

For each positive solution $u(x,t)$ of $(E), (B1)$ $[(E), (B2)]$ we associate the functions $V(t)$ and $\tilde{V}(t)$ defined by

$$V(t) = K_\Phi \int_\Omega u(x,t)\Phi(x)dx, \quad \tilde{V}(t) = \frac{1}{|\Omega|}\int_\Omega u(x,t)dx, \quad \text{and}$$

$$F(t) = \epsilon g_0 \int_a^b Q(t,\xi)d\eta(\xi),$$

where

$$K_\Phi = \left(\int_\Omega \Phi(x) dx \right)^{-1}, \qquad |\Omega| = \int_\Omega dx, \quad \text{and} \quad g_0 = 1 - c(g(t,\xi)).$$

3. Oscillation of the Problem (E) and $(B1)$

In this section, we establish sufficient conditions for the oscillation of all solutions of (E) and (B1).

Theorem 3.1. *If the impulsive functional differential inequality*

$$\left.\begin{aligned} &(r(t)Z'(t))' + p(t)Z'(t) + F(t)Z(\theta(t)) \le 0, \quad t \ne t_k \\ &a_k^* \le \frac{Z(t_k^+)}{Z(t_k)} \le a_k, \\ &b_k^* \le \frac{Z'(t_k^+)}{Z'(t_k)} \le b_k, \quad k = 1, 2, \cdots \end{aligned}\right\} \tag{1}$$

has no eventually positive solution, then every solution of the boundary value problem defined by (E) *and* $(B1)$ *is oscillatory in* G.

Proof. Assume the contrary that $u(x,t) \ne 0$ is a solution of the boundary value problem (E), $(B1)$. Which has a constant sign in the domain $\Omega \times [t_0, +\infty)$. Assume that $u(x,t) > 0$, $(x,t) \in \Omega \times [t_0, +\infty)$, $t_0 \ge 0$. By the assumption that there exists a $t_1 > t_0$ such that $g(t,\xi) \ge t_0$, $h(t,\xi) \ge t_0$ for $(t,\xi) \in [t_1, +\infty) \times [a,b]$ and $\tau(t) \ge t_0$ for $t \ge t_1$, then

$$\begin{aligned} &u(x, g(t,\xi)) > 0 \quad \text{for} \quad (x,t,\xi) \in \Omega \times [t_1, +\infty) \times [a,b], \\ &u(x, \tau(t)) > 0 \qquad \text{for} \quad (x,t) \in \Omega \times [t_1, +\infty), \\ \text{and} \quad &u(x, h(t,\xi)) > 0 \quad \text{for} \quad (x,t,\xi) \in \Omega \times [t_1, +\infty) \times [a,b]. \end{aligned}$$

For $t \geq t_0,\ t \neq t_k,\ k = 1, 2, \cdots$, multiplying both sides of equation (E) by $K_\Phi \Phi(x) > 0$ and taking integration with respect to x over the domain Ω, we obtain

$$\left.\begin{aligned}
&\frac{d}{dt}\left[r(t)\frac{d}{dt}\left(\int_\Omega u(x,t)K_\Phi\Phi(x)dx + c(t)\int_\Omega u(x,\tau(t))K_\Phi\Phi(x)dx\right)\right]\\
&+p(t)\frac{d}{dt}\left(\int_\Omega u(x,t)K_\Phi\Phi(x)dx + c(t)\int_\Omega u(x,\tau(t))K_\Phi\Phi(x)dx\right)\\
&+\int_\Omega\int_a^b q(x,t,\xi)f(u(x,g(t,\xi)))K_\Phi\Phi(x)d\eta(\xi)dx\\
&= a(t)\int_\Omega \Delta u(x,t)K_\Phi\Phi(x)dx - \int_a^b b(t,\xi)\int_\Omega \Delta u(x,h(t,\xi))K_\Phi\Phi(x)dxd\eta(\xi).
\end{aligned}\right\} \quad (2)$$

From Green's formula and boundary condition $(B1)$,

$$\begin{aligned}
K_\Phi\int_\Omega \Delta u(x,t)\Phi(x)dx = {} & K_\Phi\int_{\partial\Omega}\left[\Phi(x)\frac{\partial u}{\partial\gamma} - u\frac{\partial\Phi(x)}{\partial\gamma}\right]dS + K_\Phi\int_\Omega u(x,t)\Delta\Phi(x)dx\\
= {} & -\lambda_0 V(t) \leq 0 \qquad (3)
\end{aligned}$$

and

$$\begin{aligned}
K_\Phi\int_\Omega \Delta u(x,h(t,\xi))\Phi(x)dx = {} & K_\Phi\int_{\partial\Omega}\left[\Phi(x)\frac{\partial u(x,h(t,\xi))}{\partial\gamma} - u(x,h(t,\xi))\frac{\partial\Phi(x)}{\partial\gamma}\right]dS\\
& + K_\Phi\int_\Omega u(x,h(t,\xi))\Delta\Phi(x)dx\\
= {} & -\lambda_0 V(h(t,\xi)) \leq 0, \qquad (4)
\end{aligned}$$

where dS is surface element on $\partial\Omega$. Moreover using Jensen's inequality, from $(H2)$ and assumptions, it follows that

$$\begin{aligned}
&\int_\Omega\int_a^b q(x,t,\xi)f(u(x,g(t,\xi)))K_\Phi\Phi(x)d\eta(\xi)dx\\
&\qquad \geq \int_a^b Q(t,\xi)\int_\Omega f(u(x,g(t,\xi)))K_\Phi\Phi(x)dxd\eta(\xi)\\
&\qquad \geq \int_a^b Q(t,\xi)\epsilon\int_\Omega u(x,g(t,\xi))K_\Phi\Phi(x)dxd\eta(\xi)\\
&\qquad \geq \epsilon\int_a^b Q(t,\xi)V(g(t,\xi))d\eta(\xi). \qquad (5)
\end{aligned}$$

In view of (2)-(5), we obtain

$$\frac{d}{dt}\left[r(t)\frac{d}{dt}\left(V(t)+c(t)V(\tau(t))\right)\right]+p(t)\frac{d}{dt}\left(V(t)+c(t)V(\tau(t))\right)$$
$$+\ \epsilon\int_a^b Q(t,\xi)V(g(t,\xi))d\eta(\xi)\leq 0.$$

Set $Z(t)=V(t)+c(t)V(\tau(t))$. Then

$$(r(t)Z'(t))'+p(t)Z'(t)+\epsilon\int_a^b Q(t,\xi)V(g(t,\xi))d\eta(\xi)\leq 0,\qquad t\neq t_k. \tag{6}$$

It is easy to obtain that $Z(t)>0$ for $t\geq t_1$. Next we prove that $Z'(t)>0$ for $t\geq t_2$. Assume the contrary, there exists $T\geq t_2$ such that $Z'(T)\leq 0$.

$$(r(t)Z'(t))'+p(t)Z'(t)\leq 0,\qquad t\geq t_2$$
$$r(t)Z''(t)+\left(r'(t)+p(t)\right)Z'(t)\leq 0,\qquad t\geq t_2. \tag{7}$$

From (H_1), we have $R'(t)=R(t)\left(\dfrac{r'(t)+p(t)}{r(t)}\right)$ and $R(t)>0,\ \ R'(t)\geq 0$ for $t\geq t_2$. We multiply $\dfrac{R(t)}{r(t)}$ on both sides of (7), we have

$$R(t)Z''(t)+R'(t)Z'(t)=(R(t)Z'(t))'\leq 0,\quad t\geq t_2. \tag{8}$$

From (8), we have $R(t)Z'(t)\leq R(T)Z'(T)\leq 0,\ t\geq T$. Thus

$$\int_T^t Z'(s)ds\leq\ \int_T^t\frac{R(T)Z'(T)}{R(s)}ds,\quad t\geq T$$
$$Z(t)\leq\ Z(T)+R(T)Z'(T)\int_T^t\frac{ds}{R(s)},\quad t\geq T.$$

From the hypotheses (H_1), we have $\lim_{t\to+\infty}Z(t)=-\infty$. This contradicts that $Z(t)>0$ for $t\geq 0$. Thus $Z'(t)>0$ and $\tau(t)\leq t$ for $t\geq t_1$, we have

$$V(t)=\ Z(t)-c(t)V(\tau(t))$$
$$V(t)\geq\ Z(t)-c(t)Z(t)$$
$$V(t)\geq\ Z(t)(1-c(t))$$

and

$$
\begin{aligned}
V(g(t,\xi)) \geq&\ Z(g(t,\xi))(1-c(g(t,\xi)))\\
V(g(t,\xi)) \geq&\ g_0 Z(g(t,\xi)).
\end{aligned}
$$

Therefore from (6), we have

$$(r(t)Z'(t))' + p(t)Z'(t) + \epsilon\ g_0 \int_a^b Q(t,\xi)Z(g(t,\xi))d\eta(\xi) \leq 0, \quad t \geq t_1.$$

From (H_3) and (H_4), we have

$$Z[g(t,\xi)] \geq Z[g(t,a)] > 0, \quad \xi \in [a,b] \quad \text{and} \quad \theta(t) \leq g(t,a) \leq t,$$

thus, $Z(\theta(t)) \leq Z(g(t,a))$ for $t \geq t_1$. Therefore

$$
\begin{aligned}
(r(t)Z'(t))' + p(t)Z'(t) + \epsilon \int_a^b Q(t,\xi)Z(\theta(t))d\eta(\xi) \leq 0, &\quad t \geq t_1,\\
(r(t)Z'(t))' + p(t)Z'(t) + F(t)Z(\theta(t)) \leq 0, &\quad t \geq t_1.
\end{aligned}
$$

For $t \geq t_0,\ t = t_k,\ k = 1, 2, \cdots$, multiplying both sides of the equation (E) by $K_\Phi \Phi(x) > 0$, taking integration with respect to x over the domain Ω, and from (H_6), we obtain

$$a_k^* \leq \frac{u(x,t_k^+)}{u(x,t_k^+)} \leq a_k, \qquad b_k^* \leq \frac{u_t(x,t_k)}{u_t(x,t_k)} \leq b_k.$$

From assumptions we have,

$$a_k^* \leq \frac{V(t_k^+)}{V(t_k)} \leq a_k, \qquad b_k^* \leq \frac{V'(t_k^+)}{V'(t_k)} \leq b_k,$$

and

$$a_k^* \leq \frac{Z(t_k^+)}{Z(t_k)} \leq a_k, \qquad b_k^* \leq \frac{Z'(t_k^+)}{Z'(t_k)} \leq b_k.$$

Therefore $Z(t) > 0$ is solution of (1). This contradicts the hypothesis and completes the proof. □

Theorem 3.2. *Suppose that*

$$\limsup_{t\to+\infty} \frac{1}{t^\alpha} \mathcal{L}_{t_0}^\rho \left\{ \prod_{t_0 \leq t_k < s} \left(\frac{b_k}{a_k^*}\right)^{-1} \left[F(s) - \frac{r(\theta(s))}{4\theta'(s)} \left(\frac{p(s)}{r(s)} - \left(\frac{-\alpha}{t-s} + \frac{\rho'(s)}{\rho(s)} \right) \right)^2 \right] \right\} = +\infty \tag{9}$$

then every solution u of the boundary value problem (E), $(B1)$ is oscillatory in G.

Proof. To prove the solutions of (E), $(B1)$ are oscillatory in G, from Theorem 3.1, it is enough to prove that the impulsive functional differential inequality (1) has no eventually positive solutions. Suppose that $Z(t) > 0$ is a solution of the inequality (1). Define

$$W(t) = \frac{r(t)Z'(t)}{Z(\theta(t))}, \quad t \geq t_0.$$

Then

$$\begin{aligned} W'(t) \leq & -\frac{\theta'(t)}{r(\theta(t))}W^2(t) - \frac{p(t)}{r(t)}W(t) - F(t), \\ W(t_k^+) \leq & \frac{b_k}{a_k^*}W(t_k). \end{aligned} \tag{10}$$

Define

$$U(t) = \prod_{t_0 \leq t_k < t} \left(\frac{b_k}{a_k^*}\right)^{-1} W(t).$$

In fact, $W(t)$ is continuous on each interval $(t_k, t_{k+1}]$, and in view of $W(t_k^+) \leq \dfrac{b_k}{a_k^*}W(t_k)$. It follows that for $t \geq t_0$,

$$U(t_k^+) = \prod_{t_0 \leq t_j \leq t_k} \left(\frac{b_k}{a_k^*}\right)^{-1} W(t_k^+) \leq \prod_{t_0 \leq t_j < t_k} \left(\frac{b_k}{a_k^*}\right)^{-1} W(t_k) = U(t_k)$$

and for all $t \geq t_0$,

$$U(t_k^-) = \prod_{t_0 \leq t_j \leq t_{k-1}} \left(\frac{b_k}{a_k^*}\right)^{-1} W(t_k^-) \leq \prod_{t_0 \leq t_j < t_k} \left(\frac{b_k}{a_k^*}\right)^{-1} W(t_k) = U(t_k)$$

which implies that $U(t)$ is continuous on $[t_0, +\infty)$.

$$\begin{aligned} & U'(t) + \prod_{t_0 \leq t_k < t} \left(\frac{b_k}{a_k^*}\right) \frac{U^2(t)\theta'(t)}{r(\theta(t))} + \frac{p(t)}{r(t)}U(t) + \prod_{t_0 \leq t_k < t} \left(\frac{b_k}{a_k^*}\right)^{-1} F(t) \\ &= \prod_{t_0 \leq t_k < t} \left(\frac{b_k}{a_k^*}\right)^{-1} W'(t) + \prod_{t_0 \leq t_k < t} \left(\frac{b_k}{a_k^*}\right) \prod_{t_0 \leq t_k < t} \left(\frac{b_k}{a_k^*}\right)^{-2} \frac{\theta'(t)}{r(\theta(t))}W^2(t) \\ &\quad + \prod_{t_0 \leq t_k < t} \left(\frac{b_k}{a_k^*}\right)^{-1} \frac{p(t)}{r(t)}W(t) + \prod_{t_0 \leq t_k < t} \left(\frac{b_k}{a_k^*}\right)^{-1} F(t) \end{aligned}$$

$$= \prod_{t_0 \le t_k < t} \left(\frac{b_k}{a_k^*}\right)^{-1} \left[W'(t) + W^2(t)\frac{\theta'(t)}{r(\theta(t))} + W(t)\frac{p(t)}{r(t)} + F(t)\right] \le 0.$$

That is

$$U'(t) \le -\prod_{t_0 \le t_k < t} \left(\frac{b_k}{a_k^*}\right) \frac{\theta'(t)}{r(\theta(t))}U^2(t) - \frac{p(t)}{r(t)}U(t) - \prod_{t_0 \le t_k < t} \left(\frac{b_k}{a_k^*}\right)^{-1} F(t). \tag{11}$$

Apply the operator $\mathcal{L}_\sigma^\rho$ to (11), with t replaced by s, we get

$$\begin{aligned}
&\mathcal{L}_\sigma^\rho\left(\frac{\partial U}{\partial s}\right) \le \mathcal{L}_\sigma^\rho\left[-\prod_{t_0 \le t_k < s} \left(\frac{b_k}{a_k^*}\right) \frac{\theta'(t)}{r(\theta(t))}U^2(s) - \frac{p(s)}{r(s)}U(s) - \prod_{t_0 \le t_k < s} \left(\frac{b_k}{a_k^*}\right)^{-1} F(s)\right] \\
&- \rho(\sigma)(t-\sigma)^\alpha U(\sigma) - \mathcal{L}_\sigma^\rho\left[\left(\frac{-\alpha}{t-s} + \frac{\rho'(s)}{\rho(s)}\right)U(s)\right] \\
&\le -\mathcal{L}_\sigma^\rho\left[\prod_{t_0 \le t_k < s} \left(\frac{b_k}{a_k^*}\right) \frac{\theta'(s)}{r(\theta(s))}U^2(s) + \frac{p(s)}{r(s)}U(s) + \prod_{t_0 \le t_k < s} \left(\frac{b_k}{a_k^*}\right)^{-1} F(s)\right] \\
&- \mathcal{L}_\sigma^\rho\left[\left(\frac{-\alpha}{t-s} + \frac{\rho'(s)}{\rho(s)}\right)U(s)\right] + \mathcal{L}_\sigma^\rho\Bigg[\left(\sqrt{\prod_{t_0 \le t_k < s} \left(\frac{b_k}{a_k^*}\right) \frac{\theta'(s)}{r(\theta(s))}}U(s)\right)^2 \\
&+ 2\left(\sqrt{\prod_{t_0 \le t_k < s} \left(\frac{b_k}{a_k^*}\right) \frac{\theta'(s)}{r(\theta(s))}}U(s) \times \frac{1}{2}\frac{p(s)}{r(s)}\sqrt{\prod_{t_0 \le t_k < s} \left(\frac{b_k}{a_k^*}\right)^{-1} \frac{r(\theta(s))}{\theta'(s)}}\right) \\
&+ \frac{1}{4}\frac{p^2(s)}{r^2(s)}\prod_{t_0 \le t_k < s} \left(\frac{b_k}{a_k^*}\right)^{-1} \frac{r(\theta(s))}{\theta'(s)} - \frac{1}{4}\frac{p^2(s)}{r^2(s)}\prod_{t_0 \le t_k < s} \left(\frac{b_k}{a_k^*}\right)^{-1} \frac{r(\theta(s))}{\theta'(s)} + \prod_{t_0 \le t_k < s} \left(\frac{b_k}{a_k^*}\right)^{-1} F(s)\Bigg] \\
&\le \rho(\sigma)(t-\sigma)^\alpha U(\sigma) \\
&- \mathcal{L}_\sigma^\rho\left[\left(\frac{-\alpha}{t-s} + \frac{\rho'(s)}{\rho(s)}\right)U(s)\right] \\
&\quad + \mathcal{L}_\sigma^\rho\left[\left(\sqrt{\prod_{t_0 \le t_k < s} \left(\frac{b_k}{a_k^*}\right) \frac{\theta'(s)}{r(\theta(s))}}U(s) + \frac{1}{2}\frac{p(s)}{r(s)}\sqrt{\prod_{t_0 \le t_k < s} \left(\frac{b_k}{a_k^*}\right)^{-1} \frac{r(\theta(s))}{\theta'(s)}}\right)^2\right] \\
&+ \mathcal{L}_\sigma^\rho\left[\prod_{t_0 \le t_k < s} \left(\frac{b_k}{a_k^*}\right)^{-1} F(s) - \frac{1}{4}\frac{p^2(s)}{r^2(s)}\prod_{t_0 \le t_k < s} \left(\frac{b_k}{a_k^*}\right)^{-1} \frac{r(\theta(s))}{\theta'(s)}\right] \le \rho(\sigma)(t-\sigma)^\alpha U(\sigma).
\end{aligned} \tag{12}$$

Denote

$$Y(s)=\sqrt{\prod_{t_0\le t_k<s}\left(\frac{b_k}{a_k^*}\right)\frac{\theta'(s)}{r(\theta(s))}}U(s)+\frac{1}{2}\frac{p(s)}{r(s)}\sqrt{\prod_{t_0\le t_k<s}\left(\frac{b_k}{a_k^*}\right)^{-1}\frac{r(\theta(s))}{\theta'(s)}}.$$

Applying the above $Y(s)$ in (12), we get

$$\begin{aligned}&-\mathcal{L}_\sigma^\rho\left[\left(\frac{-\alpha}{t-s}+\frac{\rho'(s)}{\rho(s)}\right)U(s)\right]\\&\quad+\mathcal{L}_\sigma^\rho(Y^2(s))+\mathcal{L}_\sigma^\rho\left[\prod_{t_0\le t_k<s}\left(\frac{b_k}{a_k^*}\right)^{-1}F(s)-\frac{1}{4}\frac{p^2(s)}{r^2(s)}\prod_{t_0\le t_k<s}\left(\frac{b_k}{a_k^*}\right)^{-1}\frac{r(\theta(s))}{\theta'(s)}\right]\\&\quad\le\rho(\sigma)(t-\sigma)^\alpha U(\sigma)\end{aligned}$$

$$\begin{aligned}&\mathcal{L}_\sigma^\rho\left[\left(Y(s)-\frac{1}{2}\left(\frac{-\alpha}{t-s}+\frac{\rho'(s)}{\rho(s)}\right)\sqrt{\prod_{t_0\le t_k<s}\left(\frac{b_k}{a_k^*}\right)^{-1}\frac{r(\theta(s))}{\theta'(s)}}\right)^2\right]\\&+\mathcal{L}_\sigma^\rho\left[\prod_{t_0\le t_k<s}\left(\frac{b_k}{a_k^*}\right)^{-1}F(s)-\prod_{t_0\le t_k<s}\left(\frac{b_k}{a_k^*}\right)^{-1}\frac{r(\theta(s))}{4\theta'(s)}\left(\frac{p(s)}{r(s)}-\left(\frac{-\alpha}{t-s}+\frac{\rho'(s)}{\rho(s)}\right)\right)^2\right]\\&\le\rho(\sigma)(t-\sigma)^\alpha U(\sigma).\end{aligned}\tag{13}$$

Note that, the first term of (13) is nonnegative, so

$$\begin{aligned}&\mathcal{L}_\sigma^\rho\left[\prod_{t_0\le t_k<s}\left(\frac{b_k}{a_k^*}\right)^{-1}F(s)-\prod_{t_0\le t_k<s}\left(\frac{b_k}{a_k^*}\right)^{-1}\frac{r(\theta(s))}{4\theta'(s)}\left(\frac{p(s)}{r(s)}-\left(\frac{-\alpha}{t-s}+\frac{\rho'(s)}{\rho(s)}\right)\right)^2\right]\\&\qquad\le\rho(\sigma)t^\alpha\left(1-\frac{\sigma}{t}\right)^\alpha U(\sigma),\quad t\ge s\ge t_0.\end{aligned}\tag{14}$$

Let $\sigma=t_0$ and taking lim sup in (14) as $t\to+\infty$, we get

$$\begin{aligned}&\limsup_{t\to+\infty}\frac{1}{t^\alpha}\mathcal{L}_{t_0}^\rho\left\{\prod_{t_0\le t_k<s}\left(\frac{b_k}{a_k^*}\right)^{-1}\left[F(s)-\frac{r(\theta(s))}{4\theta'(s)}\left(\frac{p(s)}{r(s)}-\left(\frac{-\alpha}{t-s}+\frac{\rho'(s)}{\rho(s)}\right)\right)^2\right]\right\}\\&\qquad\le\rho(t_0)U(t_0)<+\infty,\end{aligned}\tag{15}$$

which is a contradiction to (9). This completes the proof. □

Corollary 3.3. *If* (9) *in Theorem 3.2 is replaced by*

$$\limsup_{t\to+\infty}\frac{1}{t^\alpha}\mathcal{L}_{t_0}^\rho\left[\prod_{t_0\le t_k<s}\left(\frac{b_k}{a_k^*}\right)^{-1}F(s)\right]=+\infty$$

and

$$\limsup_{t\to+\infty} \frac{1}{t^\alpha}\mathcal{L}_{t_0}^{\rho}\left[\prod_{t_0\le t_k<s}\left(\frac{b_k}{a_k^*}\right)^{-1}\frac{r(\theta(s))}{4\theta'(s)}\left(\frac{p(s)}{r(s)}-\left(\frac{-\alpha}{t-s}+\frac{\rho'(s)}{\rho(s)}\right)\right)^2\right] < +\infty,$$

then every solution u of (E), $(B1)$ is oscillatory in G.

4. Oscillation of the Problem (E) and $(B2)$

In this section, we investigate the oscillation of the problem (E) and $(B2)$.

Theorem 4.1. *If the impulsive functional differential inequality*

$$\left.\begin{array}{l} (r(t)\tilde{Z}'(t))' + p(t)\tilde{Z}'(t) + F(t)\tilde{Z}(\theta(t)) \le 0, \quad t \ne t_k \\ a_k^* \le \dfrac{\tilde{Z}(t_k^+)}{\tilde{Z}(t_k)} \le a_k, \\ b_k^* \le \dfrac{\tilde{Z}'(t_k^+)}{\tilde{Z}'(t_k)} \le b_k, \quad k = 1, 2, \cdots \end{array}\right\} \tag{16}$$

has no eventually positive solution, then every solution of the boundary value problem defined by (E) and $(B2)$ is oscillatory in G.

Proof. Suppose to the contrary that $u(x,t) \ne 0$ is solution of the boundary value problem (E), $(B2)$. Which has a constant sign in the domain $\Omega\times[t_0,+\infty)$. Assume that $u(x,t) > 0$, $(x,t) \in \Omega\times[t_0,+\infty)$, $t_0 \ge 0$. By the assumption that there exists a $t_1 > t_0$ such that $g(t,\xi) \ge t_0$, $h(t,\xi) \ge t_0$ for $(t,\xi) \in [t_1,+\infty)\times[a,b]$ and $\tau(t) \ge t_0$ for $t \ge t_1$, then

$$\begin{aligned} & u(x,g(t,\xi)) > 0 \quad \text{for} \quad (x,t,\xi) \in \Omega\times[t_1,+\infty)\times[a,b], \\ & u(x,\tau(t)) > 0 \quad \text{for} \quad (x,t) \in \Omega\times[t_1,+\infty), \\ \text{and} \quad & u(x,h(t,\xi)) > 0 \quad \text{for} \quad (x,t,\xi) \in \Omega\times[t_1,+\infty)\times[a,b]. \end{aligned}$$

For $t \ge t_0$, $t \ne t_k$, $k = 1, 2, \cdots$, multiplying both sides of equation (E) by $\dfrac{1}{|\Omega|}$ and integrating with respect to x over the domain Ω, we have

$$\left.\begin{aligned}
&\frac{d}{dt}\left[r(t)\frac{d}{dt}\left(\frac{1}{|\Omega|}\int_\Omega u(x,t)dx+\frac{1}{|\Omega|}c(t)\int_\Omega u(x,\tau(t))dx\right)\right]\\
&+p(t)\frac{d}{dt}\left(\frac{1}{|\Omega|}\int_\Omega u(x,t)dx+\frac{1}{|\Omega|}c(t)\int_\Omega u(x,\tau(t))dx\right)\\
&+\frac{1}{|\Omega|}\int_\Omega\int_a^b q(x,t,\xi)f(u(x,g(t,\xi)))d\eta(\xi)dx\\
&=a(t)\frac{1}{|\Omega|}\int_\Omega \Delta u(x,t)dx-\int_a^b b(t,\xi)\frac{1}{|\Omega|}\int_\Omega \Delta u(x,h(t,\xi))dxd\eta(\xi).
\end{aligned}\right\} \tag{17}$$

From Green's formula and boundary condition $(B2)$, yield

$$\int_\Omega \Delta u(x,t)dx = \int_{\partial\Omega}\frac{\partial u}{\partial\gamma}dS = -\int_{\partial\Omega}\mu(x,t)u(x,t)dS \le 0, \tag{18}$$

and

$$\begin{aligned}
\int_\Omega \Delta u(x,h(t,\xi))dx &= \int_{\partial\Omega}\frac{\partial u(x,h(t,\xi))}{\partial\gamma}dS\\
&= -\int_{\partial\Omega}\mu(x,h(t,\xi))u(x,h(t,\xi))dS \le 0,
\end{aligned} \tag{19}$$

where dS is the surface element on $\partial\Omega$. Also from (H_2) and Jensen's inequality, we have

$$\begin{aligned}
&\frac{1}{|\Omega|}\int_\Omega\int_a^b q(x,t,\xi)f(u(x,g(t,\xi)))d\eta(\xi)dx\\
&\qquad \ge \int_a^b Q(t,\xi)\frac{1}{|\Omega|}\int_\Omega f(u(x,g(t,\xi)))dxd\eta(\xi)\\
&\qquad \ge \int_a^b Q(t,\xi)\ \epsilon\ \frac{1}{|\Omega|}\int_\Omega u(x,g(t,\xi))dxd\eta(\xi)\\
&\qquad \ge \int_a^b Q(t,\xi)\epsilon\ \tilde{V}(g(t,\xi))d\eta(\xi).
\end{aligned} \tag{20}$$

In view of (17)-(20), yield

$$\begin{aligned}
&\frac{d}{dt}\left[r(t)\frac{d}{dt}\left(\tilde{V}(t)+c(t)\tilde{V}(\tau(t))\right)\right]+p(t)\frac{d}{dt}\left(\tilde{V}(t)+c(t)\tilde{V}(\tau(t))\right)\\
&\qquad\qquad +\epsilon\int_a^b Q(t,\xi)\tilde{V}(g(t,\xi))d\eta(\xi)\le 0.
\end{aligned}$$

Set $\tilde{Z}(t) = \tilde{V}(t) + c(t)\tilde{V}(\tau(t))$. Then

$$\left(r(t)\tilde{Z}'(t)\right)' + p(t)\tilde{Z}'(t) + \epsilon \int_a^b Q(t,\xi)\tilde{V}(g(t,\xi))d\eta(\xi) \leq 0, \quad t \neq t_k. \tag{21}$$

Rest of the proof is similar to Theorem 3.1, and therefore we omit it. □

Theorem 4.2. *Suppose that*

$$\limsup_{t\to+\infty} \frac{1}{t^\alpha}\mathcal{L}^\rho_{t_0}\left\{\prod_{t_0\leq t_k<s}\left(\frac{b_k}{a^*_k}\right)^{-1}\left[F(s)-\frac{r(\theta(s))}{4\theta'(s)}\left(\frac{p(s)}{r(s)}-\left(\frac{-\alpha}{t-s}+\frac{\rho'(s)}{\rho(s)}\right)\right)^2\right]\right\} = +\infty \tag{22}$$

then every solution u of the boundary value problem (E), $(B2)$ is oscillatory in G.

Proof. The proof is similar to that of Theorem 3.2 and therefore the details are omitted. □

Corollary 4.3. *If* (22) *in Theorem 4.2 is replaced by*

$$\limsup_{t\to+\infty} \frac{1}{t^\alpha}\mathcal{L}^\rho_{t_0}\left[\prod_{t_0\leq t_k<s}\left(\frac{b_k}{a^*_k}\right)^{-1} F(s)\right] = +\infty$$

and

$$\limsup_{t\to+\infty} \frac{1}{t^\alpha}\mathcal{L}^\rho_{t_0}\left[\prod_{t_0\leq t_k<s}\left(\frac{b_k}{a^*_k}\right)^{-1} \frac{r(\theta(s))}{4\theta'(s)}\left(\frac{p(s)}{r(s)}-\left(\frac{-\alpha}{t-s}+\frac{\rho'(s)}{\rho(s)}\right)\right)^2\right] < +\infty,$$

then every solution u of (E), $(B2)$ is oscillatory in G.

5. Example

In this section, we will present an example to illustrate the main results.

Example 5.1. *Consider the following equation of the form*

$$\left.\begin{aligned}
&\frac{\partial}{\partial t}\left[2\frac{\partial}{\partial t}\left(u(x,t)+\frac{1}{2}u(x,t-\pi)\right)\right]+\left(-\frac{8}{5}\right)\frac{\partial}{\partial t}\left(u(x,t)+\frac{1}{2}u(x,t-\pi)\right)\\
&+\frac{4}{5}\int_{-\pi/2}^{-\pi/4} u(x,t+2\xi)d\xi = \frac{1}{5}\Delta u(x,t) - \frac{12}{5}\int_{-\pi/2}^{-\pi/4}\Delta u(x,t+2\xi)d\xi,\\
&\qquad\qquad\qquad\qquad t>1,\ t\neq 2^k,\ k=1,2,\cdots,\\
&u(x,(2^k)^+) = \frac{k+1}{k}u(x,2^k),\\
&u_t(x,(2^k)^+) = u_t(x,2^k), \quad k=1,2,\cdots
\end{aligned}\right\} \tag{23}$$

for $(x,t) \in (0,\pi) \times \mathbb{R}^+$, *with the boundary condition*

$$u(0,t) = u(\pi,t) = 0, \quad t \neq 2^k. \tag{24}$$

Here $\Omega = (0,\pi)$, $a_k = a_k^* = \dfrac{k+1}{k}$, $b_k = b_k^* = 1$, $r(t) = 2$, $c(t) = \dfrac{1}{2}$, $\tau(t) = t - \pi$, $p(t) = -\dfrac{8}{5}$, $Q(t,\xi) = \dfrac{4}{5}$, $g(t,\xi) = h(t,\xi) = t + 2\xi$, $a(t) = \dfrac{1}{5}$, $b(t,\xi) = \dfrac{12}{5}$, $\eta(\xi) = \xi$, $\alpha = 2$, $\theta(t) = t$, $\rho(t) = 2$, $\epsilon = 1$. *Since* $t_0 = 1$, $t_k = 2^k$, $g_0 = 1 - c(g(t,\xi)) = \dfrac{1}{2}$, $F(t) = 1 \times \dfrac{1}{2} \times \int_{-\pi/2}^{-\pi/4} \dfrac{4}{5} d\xi = \dfrac{\pi}{10}$. *Then hypotheses* $(H_1) - (H_6)$ *hold, moreover*

$$\begin{aligned}
\lim_{t\to+\infty} \int_{t_0}^{t} \prod_{t_0 \leq t_k < s} \frac{a_k^*}{b_k} ds &= \int_1^{+\infty} \prod_{1 < t_k < s} \frac{k}{k+1} ds \\
&= \int_1^{t_1} \prod_{1 < t_k < s} \frac{k}{k+1} ds + \int_{t_1^+}^{t_2} \prod_{1 < t_k < s} \frac{k}{k+1} ds + \int_{t_2^+}^{t_3} \prod_{1 < t_k < s} \frac{k}{k+1} ds + \cdots \\
&= 1 + \frac{1}{2} \times 2 + \frac{1}{2} \times \frac{2}{3} \times 2^2 + \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times 2^3 + \cdots \\
&= \sum_{n=0}^{+\infty} \frac{2^n}{n+1} = +\infty.
\end{aligned}$$

Thus,

$$\limsup_{t\to+\infty} \frac{1}{t^2} \left\{ \int_1^t \prod_{1 < t_k < s} \frac{k}{k+1} \left((t-s)^2 \left[\frac{\pi}{5} - \left(-\frac{4}{5} + \frac{2}{t-s} \right)^2 \right] \right) ds \right\} = +\infty.$$

Hence (9) *holds. Therefore all the conditions of the Theorem 3.2 are satisfied. Therefore, every solution of equation* (23)-(24) *is oscillatory in* $\Omega \times \mathbb{R}^+$. *In fact* $u(x,t) = \sin x \cos t$ *is such a solution.*

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