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Oscillation of Even Order Impulsive Neutral Partial Differential Equations with Distributed Deviating Arguments

[1]V. Sadhasivam, [2]T. Kalaimani and [3]T. Raja

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Abstract. In this paper, we will consider a class of boundary value problems associated with even order nonlinear impulsive neutral partial functional differential equations with continuous distributed deviating arguments. Sufficient conditions are obtained for the oscillation of solutions using impulsive differential inequalities and integral averaging scheme with pair of boundary conditions. Examples are specified to point up our important results.

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Key Words: Neutral partial differential equations, Oscillation, Impulse, Distributed deviating arguments.

1. Introduction

The oscillation theory of ordinary differential equations marks its initiation through the research article of C.Sturm [19] in 1836 and for partial differential equations by P.Hartman and A.Wintner [6] in 1955. In 1989, the early work on impulsive delay differential equations [3] was published and its results were included in monograph [9]. After two years the most important exertion concluded in [2] on impulsive partial differential equations in 1991. Numerous substantial phenomena are articulated in terms of second order equations. The theoretical background of the second and even order equations are nearly common and for this reason, we study the even order equations. Impulsive ordinary and partial functional differential equations have wide range of applications in a variety of fields of science and machinery [1, 8, 18, 24].

[1]Corresponding Author: E-mail: ovsadha@gmail.com

[1,2,3]Department of Mathematics, Thiruvalluvar Government Arts College, Rasipuram, Namakkal (Dt), Tamil Nadu, India - 637 401

The oscillation of impulsive and non-impulsive parabolic and hyperbolic equations has been widely studied in the literature [13, 15, 16, 17, 20, 21, 25]. Curiously very few significant consequences on higher order partial differential equations with continuous distributed deviating arguments have been studied in [4, 10, 11, 12, 23]. But these are not considered with impulsive force. Consequently, it is necessary to study with impulse effect on the oscillation of higher order partial differential equations. To the best of authors' acquaintance, there are no theoretical results on the oscillation of higher order nonlinear impulsive neutral partial differential equations with continuous distributed deviating arguments. In this fashion, we initiate oscillatory results for even order nonlinear impulsive neutral partial differential equations with continuous distributed deviating arguments of the type $(E),(B_1)[(E),(B_2)]$. Focal results of this manuscript expand and improve numerous findings in the earlier publications of non-impulse type equations. We think likely that this primary work attain the absorption of numerous researchers working on the even order impulsive partial functional differential equations.

In this work, we focus on the following even order nonlinear impulsive neutral partial functional differential equation with continuous distributed deviating arguments

$$\left.\begin{array}{l}
\dfrac{\partial^m}{\partial t^m}\left[u(x,t)+c(t)u(x,\tau(t))\right]+\int_a^b q(x,t,\xi)f(u(x,\sigma(t,\xi)))d\eta(\xi) \\
= a(t)\Delta u(x,t)-\int_a^b b(t,\xi)\Delta u(x,\rho(t,\xi))d\eta(\xi), \quad t\neq t_k,\ (x,t)\in\Omega\times(0,+\infty)\equiv G \\
\dfrac{\partial^{(i)}u(x,t_k^+)}{\partial t^{(i)}} = I_k^{(i)}\left(x,t_k,\dfrac{\partial^{(i)}u(x,t_k)}{\partial t^{(i)}}\right), \quad t=t_k, \quad k=1,2,\cdots,i=0,1,2,\cdots,m-1
\end{array}\right\} \tag{E}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a piecewise smooth boundary $\partial\Omega$ and Δ is the Laplacian in the Euclidean space $\mathbb{R}^N$.

Equation (E) is enhancement with one of the subsequent Dirichlet and Robin boundary conditions,

$$u = 0, \qquad (x,t) \in \partial\Omega \times (0,+\infty) \tag{B_1}$$

$$\frac{\partial u}{\partial \gamma} + \mu(x,t)u = 0, \qquad (x,t) \in \partial\Omega \times (0,+\infty) \tag{B_2}$$

where γ is the outer surface normal vector to $\partial\Omega$ and $\mu(x,t) \in C\left(\partial\Omega \times [0,+\infty), [0,+\infty)\right)$.

In the sequel, we assume that the following hypotheses (H) hold:

(H_1) $a(t) \in PC\left([0,+\infty), [0,+\infty)\right)$, where PC represents the class of functions which are piecewise continuous in t with discontinuities of first kind only at $t = t_k,\ k = 1,2,\cdots,$ and left continuous at $t = t_k,\ k = 1,2,\cdots,$ $\tau(t) \in C([0,+\infty),\mathbb{R})$ and $\lim\limits_{t\to+\infty} \tau(t) = +\infty$.

(H_2) $c(t) \in C^m([0,+\infty),[0,+\infty))$, $q(x,t,\xi) \in C(\bar{\Omega} \times [0,+\infty) \times [a,b], [0,+\infty))$, $Q(t,\xi) = \min\limits_{x\in\bar{\Omega}} q(x,t,\xi)$, $b(t,\xi) \in C([0,+\infty) \times [a,b], [0,+\infty))$, $f(u) \in C(\mathbb{R},\mathbb{R})$ is convex in $[0,+\infty)$, $uf(u) > 0$ and $\dfrac{f(u)}{u} \geq \epsilon > 0$ for $u \neq 0$.

(H_3) $\sigma(t,\xi),\ \rho(t,\xi) \in C([0,+\infty) \times [a,b], \mathbb{R})$, $\sigma(t,\xi) \leq t$, $\rho(t,\xi) \leq t$ for $\xi \in [a,b]$, $\sigma(t,\xi)$ and $\rho(t,\xi)$ are nondecreasing with respect to t and ξ respectively and $\liminf\limits_{t\to+\infty,\ \xi\in[a,b]} \sigma(t,\xi) = \liminf\limits_{t\to+\infty,\ \xi\in[a,b]} \rho(t,\xi) = +\infty$, a,b are non-positive constants with $a < b$.

(H_4) There exists a function $\theta(t) \in C([0,+\infty),[0,+\infty))$ satisfying $\theta(t) \leq \sigma(t,a)$, $\theta'(t) > 0$ and $\lim\limits_{t\to+\infty} \theta(t) = +\infty$, $\eta(\xi) : [a,b] \to \mathbb{R}$ is nondecreasing and the integral is a Stieltjes integral in (E).

(H_5) $\dfrac{\partial^{(i)} u(x,t)}{\partial t^{(i)}}$ are piecewise continuous in t with discontinuities of first kind only at $t = t_k,\ k = 1,2,\cdots,$ and left continuous at $t = t_k$, $\dfrac{\partial^{(i)} u(x,t_k)}{\partial t^{(i)}} = \dfrac{\partial^{(i)} u(x,t_k^-)}{\partial t^{(i)}}$, $k = 1,2,\cdots,$ $i = 0,1,2,\cdots,m-1$.

(H_6) $I_k^{(i)}\left(x, t_k, \dfrac{\partial^{(i)} u(x,t_k)}{\partial t^{(i)}}\right) \in PC(\bar{\Omega} \times [0,+\infty) \times \mathbb{R}, \mathbb{R})$, $k = 1,2,\cdots,$ $i = 0,1,2,\cdots,m-1$, and there exist positive constants $a_k^{(i)}, b_k^{(i)}$ with

$b_k^{(m-1)} \leq a_k^{(0)}$ such that for $i = 0, 1, 2, \cdots, m-1,\ k = 1, 2, \cdots,$

$$a_k^{(i)} \leq \frac{I_k^{(i)}\left(x, t_k, \dfrac{\partial^{(i)} u(x,t_k)}{\partial t^{(i)}}\right)}{\dfrac{\partial^{(i)} u(x,t_k)}{\partial t^{(i)}}} \leq b_k^{(i)}.$$

This paper is considered as follows: Section 2, presents the definitions and notations. In section 3, we deal with the oscillation of the problem (E) and (B_1). In section 4, we discuss the oscillation of the problem (E) and (B_2). Section 5, presents examples to illustrate the main results.

2. Preliminaries

In this section, we begin with definitions and known results which are required throughout this paper.

Definition 2.1. A solution u of the problem (E) is a function $u \in C^m(\bar{\Omega} \times [t_{-1}, +\infty), \mathbb{R}) \cap C(\bar{\Omega} \times [\hat{t}_{-1}, +\infty), \mathbb{R})$ that satisfies (E), where

$$t_{-1} := \min\left\{0, \inf_{t\geq 0} \tau(t)\right\} \quad \text{and}$$

$$\hat{t}_{-1} := \min\left\{0, \min_{\xi\in[a,b]}\left\{\inf_{t\geq 0}\sigma(t,\xi)\right\}, \min_{\xi\in[a,b]}\left\{\inf_{t\geq 0}\rho(t,\xi)\right\}\right\}.$$

Definition 2.2. The solution u of the problem $(E), (B_1)$ $[(E), (B_2)]$ is said to be oscillatory in the domain G if for any positive number ℓ there exist a point $(x_0, t_0) \in \Omega \times [\ell, +\infty)$ such that $u(x_0, t_0) = 0$ holds.

Definition 2.3. A function $V(t)$ is said to be eventually positive (negative) if there exists a $t_1 \geq t_0$ such that $V(t) > 0$ (< 0) holds for all $t \geq t_1$.

It is identified that [22] the least eigenvalue $\lambda_0 > 0$ of the eigenvalue problem

$$\Delta\omega(x) + \lambda\omega(x) = 0 \quad \text{in} \quad \Omega$$
$$\omega(x) = 0 \quad \text{on} \quad \partial\Omega$$

and the consequent eigenfunction $\Phi(x) > 0$ in Ω.

For each positive solution $u(x,t)$ of the problem $(E),(B_1)$ $[(E),(B_2)]$ we combine the functions $V(t)$ and $\tilde{V}(t)$ defined by

$$V(t) = K_\Phi \int_\Omega u(x,t)\Phi(x)dx, \qquad \tilde{V}(t) = \frac{1}{|\Omega|}\int_\Omega u(x,t)dx,$$
$$F(t) = M(\theta(t))^{m-2}\theta'(t), \quad \text{and} \quad G(t) = \epsilon g_0 \int_a^b Q(t,\xi)d\eta(\xi)$$

where

$$K_\Phi = \left(\int_\Omega \Phi(x)dx\right)^{-1}, \qquad |\Omega| = \int_\Omega dx, \quad \text{and} \quad g_0 = 1 - c(\sigma(t,\xi)).$$

Lemma 2.4. [7] *Let $y(t)$ be a positive and n times differentiable function on $[0,+\infty)$. If $y^{(n)}(t)$ is constant sign and not identically zero on any ray $[t_1,+\infty)$ for $t_1 > 0$, then there exists a $t_y \geq t_1$ and integer l $(0 \leq l \leq n)$, with $n+l$ even for $y(t)y^{(n)}(t) \geq 0$ or $n+l$ odd for $y(t)y^{(n)}(t) \leq 0$; and for $t \geq t_y$, $y(t)y^{(k)}(t) > 0$, $\quad 0 \leq k \leq l$; $(-1)^{k-l}y(t)y^{(k)}(t) > 0$, $\quad l \leq k \leq n$.*

Lemma 2.5. [14] *Suppose that the conditions of Lemma 2.4 is satisfied, and*

$$y^{(n-1)}(t)y^{(n)}(t) \leq 0, \quad t \geq t_y.$$

Then there exist constant $\alpha \in (0,1)$ and $M > 0$ such that for sufficiently large t

$$|y'(\alpha t)| \geq M t^{n-2} \left|y^{(n-1)}(t)\right|.$$

Lemma 2.6. [5] *If X and Y are nonnegative, then*

$$X^\alpha - \alpha X Y^{\alpha-1} + (\alpha-1)Y^\alpha \geq 0, \quad \alpha > 1$$
$$X^\alpha - \alpha X Y^{\alpha-1} - (1-\alpha)Y^\alpha \leq 0, \quad 0 < \alpha < 1,$$

where the equality holds if and only if $X = Y$.

3. Oscillation of the Problem (E) and (B_1)

In this section, we establish sufficient conditions for the oscillation of all solutions of the problem (E), (B_1).

Lemma 3.1. *If the functional impulsive differential inequality*

$$\left.\begin{array}{l} Z^{(m)}(t)+G(t)Z(\theta(t))\leq 0, \quad t\neq t_k \\ a_k^{(i)}\leq \dfrac{\dfrac{\partial^{(i)}Z(t_k^+)}{\partial t^{(i)}}}{\dfrac{\partial^{(i)}Z(t_k)}{\partial t^{(i)}}}\leq b_k^{(i)}, \quad k=1,2,\cdots, \quad i=0,1,2,\cdots,m-1 \end{array}\right\} \tag{1}$$

has no eventually positive solution, then every solution of the boundary value problem defined by (E) *and* (B_1) *is oscillatory in* G.

Proof. Assume that there exist a nonoscillatory solution $u(x,t)$ of the boundary value problem (E), (B_1) and $u(x,t)>0$. By the hypothesis (H_1) and (H_3), that there exists a $t_1>t_0>0$ such that $\tau(t)\geq t_0$, $\sigma(t,\xi),\rho(t,\xi)\geq t_0$ for $(t,\xi)\in[t_1,+\infty)\times[a,b]$ for $t\geq t_1$, then

$$\begin{aligned} &u(x,\tau(t))>0 \quad \text{for} \quad (x,t)\in\Omega\times[t_1,+\infty),\\ &u(x,\sigma(t,\xi))>0 \quad \text{for} \quad (x,t,\xi)\in\Omega\times[t_1,+\infty)\times[a,b]\\ \text{and}\quad &u(x,\rho(t,\xi))>0 \quad \text{for} \quad (x,t,\xi)\in\Omega\times[t_1,+\infty)\times[a,b]. \end{aligned}$$

For $t\geq t_0,\ t\neq t_k,\ k=1,2,\cdots$, multiplying both sides of equation (E) by $K_\Phi\Phi(x)>0$ and integrating with respect to x over the domain Ω, we attain

$$\left.\begin{array}{l} \dfrac{d^m}{dt^m}\left[\int_\Omega u(x,t)K_\Phi\Phi(x)dx+\int_\Omega c(t)u(x,\tau(t))K_\Phi\Phi(x)dx\right]\\ +\int_\Omega\int_a^b q(x,t,\xi)f(u(x,\sigma(t,\xi)))K_\Phi\Phi(x)d\eta(\xi)dx\\ =a(t)\int_\Omega\Delta u(x,t)K_\Phi\Phi(x)dx-\int_\Omega\int_a^b b(t,\xi)\Delta u(x,\rho(t,\xi))K_\Phi\Phi(x)d\eta(\xi)dx. \end{array}\right\} \tag{2}$$

From Green's formula and boundary condition (B_1), we see that

$$\begin{aligned} K_\Phi\int_\Omega\Delta u(x,t)\Phi(x)dx &= K_\Phi\int_{\partial\Omega}\left[\Phi(x)\frac{\partial u}{\partial\gamma}-u\frac{\partial\Phi(x)}{\partial\gamma}\right]dS+K_\Phi\int_\Omega u(x,t)\Delta\Phi(x)dx\\ &= -\lambda_0V(t)\leq 0 \end{aligned} \tag{3}$$

and

$$\begin{aligned}
K_\Phi \int_\Omega \Delta u(x,\rho(t,\xi))\Phi(x)dx = {} & K_\Phi \int_{\partial\Omega} \left[\Phi(x)\frac{\partial u(x,\rho(t,\xi))}{\partial\gamma} - u(x,\rho(t,\xi))\frac{\partial\Phi(x)}{\partial\gamma}\right] dS \\
& + K_\Phi \int_\Omega u(x,\rho(t,\xi))\Delta\Phi(x)dx \\
= {} & -\lambda_0 V(\rho(t,\xi)) \leq 0, \qquad (4)
\end{aligned}$$

where dS is surface component on $\partial\Omega$. Furthermore applying Jensen's inequality for convex functions and using the assumptions on (H_2), we get that

$$\begin{aligned}
& \int_\Omega \int_a^b q(x,t,\xi) f(u(x,\sigma(t,\xi))) K_\Phi \Phi(x) d\eta(\xi) dx \\
& \geq \int_a^b Q(t,\xi) \int_\Omega f(u(x,\sigma(t,\xi))) K_\Phi \Phi(x) dx d\eta(\xi) \\
& = \int_a^b Q(t,\xi)\ \epsilon \int_\Omega u(x,\sigma(t,\xi)) K_\Phi \Phi(x) dx d\eta(\xi) \\
& \geq \epsilon \int_a^b Q(t,\xi) V(\sigma(t,\xi)) d\eta(\xi). \qquad (5)
\end{aligned}$$

In consideration of (2)-(5), we acquire

$$\frac{d^m}{dt^m}\left[V(t) + c(t)V(\tau(t))\right] + \epsilon \int_a^b Q(t,\xi)V(\sigma(t,\xi))d\eta(\xi) \leq 0. \qquad (6)$$

Set $Z(t) = V(t) + c(t)V(\tau(t))$. Equation (6), can be written as

$$Z^{(m)}(t) + \epsilon \int_a^b Q(t,\xi)V(\sigma(t,\xi))d\eta(\xi) \leq 0, \quad t \neq t_k. \qquad (7)$$

From the assumption of $c(t)$ and $Q(t,\xi)$, we have $Z(t) \geq V(t) > 0$ and $Z^{(m)}(t) \leq 0$. Simultaneously, we can further prove $Z^{(m-1)}(t) \geq 0$, $t \geq t_2$. In addition, from Lemma 2.4, there exists a $t_3 \geq t_2$ and a odd number l, $0 \leq l \leq m-1$, and for $t \geq t_3$, we have

$$\begin{aligned}
Z^{(i)}(t) > 0, & \qquad 0 \leq i \leq l, \\
(-1)^{(i-1)} Z^{(i)}(t) > 0, & \qquad l \leq i \leq m-1.
\end{aligned}$$

By choosing $i = 1$, we have $Z'(t) > 0$, since $Z(t) \geq x(t) > 0$, $Z'(t) \geq 0$, we have

$$Z(\sigma(t,\xi)) \geq Z(\sigma(t,\xi) - \tau(t,\xi)) \geq x(\sigma(t,\xi) - \tau(t,\xi)),$$

and thus

$$Z^{(m)}(t) + \epsilon \int_a^b Q(t,\xi) Z(\sigma(t,\xi)) \left(1 - c(\sigma(t,\xi))\right) d\eta(\xi) \leq 0. \tag{8}$$

From equation (7), we get

$$Z^{(m)}(t) + G(t) Z(\sigma(t,\xi)) \leq 0.$$

From (H_3) and (H_4), we have

$$Z(\sigma(t,\xi)) \geq Z(\sigma(t,a)) > 0, \quad \xi \in [a,b] \ \text{ and } \quad \theta(t) \leq \sigma(t,\xi) \leq t.$$

Thus $Z(\theta(t)) \leq Z(\sigma(t,a))$ for $t \geq t_2$. Then (3.8) can be written as

$$Z^{(m)}(t) + G(t) Z(\theta(t)) \leq 0.$$

For $t \geq t_0,\ t = t_k,\ k = 1, 2, \cdots,\ i = 0, 1, 2, \cdots, m-1$, multiplying both sides of the equation (E) by $K_\Phi \Phi(x) > 0$, integrating with respect to x over the domain Ω, and from (H_6), we obtain

$$a_k^{(i)} \leq \frac{\dfrac{\partial^{(i)} u(x,t_k^+)}{\partial t^{(i)}}}{\dfrac{\partial^{(i)} u(x,t_k)}{\partial t^{(i)}}} \leq b_k^{(i)}.$$

According to $V(t) = K_\Phi \int_\Omega u(x,t)\Phi(x)dx$, we have

$$a_k^{(i)} \leq \frac{\dfrac{\partial^{(i)} V(x,t_k^+)}{\partial t^{(i)}}}{\dfrac{\partial^{(i)} V(x,t_k)}{\partial t^{(i)}}} \leq b_k^{(i)}.$$

Since $Z(t) = V(t) + c(t)V(\tau(t))$, we obtain

$$a_k^{(i)} \leq \frac{\dfrac{\partial^{(i)} Z(x,t_k^+)}{\partial t^{(i)}}}{\dfrac{\partial^{(i)} Z(x,t_k)}{\partial t^{(i)}}} \leq b_k^{(i)}.$$

Therefore $Z(t)$ is an eventually positive solution of (1). This disagree with the hypothesis. $\square$

Theorem 3.2. *If there exists a function $\varphi(t) \in C^1([0,+\infty),(0,+\infty))$ which is nondecreasing with respect to t, such that*

$$\int_{t_1}^{+\infty} \prod_{t_0 \le t_k < s} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}} \right)^{-1} \left[\varphi(s)G(s) - \frac{(\varphi'(s))^2}{4F(s)\varphi(s)} \right] ds = +\infty, \tag{9}$$

then every solution of the boundary value problem (E) and (B_1) is oscillatory in G.

Proof. Assume that there exists a nonoscillatory solution $u(x,t)$ of the boundary value problem (E), (B_1) and $u(x,t) > 0$. Proceeding as in the proof of Lemma 3.1 to get that

$$Z^{(m)}(t) + G(t)Z(\theta(t)) \le 0,$$

where $Z(t) = V(t) + c(t)V(\tau(t))$ and satisfies $Z^{(m)}(t) \le 0$, $Z^{(m-1)}(t) \ge 0$ and an odd number l, $0 \le l \le m-1$, such that

$$Z^{(i)}(t) > 0, \ 0 \le i \le l, \quad (-1)^{(i-1)} Z^{(i)}(t) > 0, \ \text{for } l \le i \le m-1.$$

Define

$$W(t) := \varphi(t) \frac{Z^{(m-1)}(t)}{Z(\theta(t))}, \quad t \ge t_0,$$

then $W(t) \ge 0$ for $t \ge t_1$, and

$$W'(t) \le \frac{\varphi'(t)}{\varphi(t)} W(t) + \frac{\varphi(t)Z^{(m)}(t)}{Z(\theta(t))} - \frac{\varphi(t)Z^{(m-1)}(t)Z'(\theta(t))\theta'(t)}{Z(\theta(t))^2}.$$

From $Z^{(m)}(t) \le 0$, according to Lemma 2.5, we obtain

$$Z'(\theta(t)) \ge M(\theta(t))^{m-2} Z^{(m-1)}(t).$$

Thus

$$\begin{aligned} W'(t) &\le \frac{\varphi'(t)}{\varphi(t)} W(t) - G(t)\varphi(t) - \frac{F(t)}{\varphi(t)} W^2(t) \\ W(t_k^+) &\le \frac{b_k^{(m-1)}}{a_k^{(0)}} W(t_k). \end{aligned}$$

Define

$$U(t) = \prod_{t_0 \le t_k < t} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}} \right)^{-1} W(t).$$

In fact, $W(t)$ is continuous on each interval $(t_k, t_{k+1}]$, and in consideration of $W(t_k^+) \leq \left(b_k^{(m-1)}/a_k^{(0)}\right) W(t_k)$. It follows for $t \geq t_0$ that

$$U(t_k^+) = \prod_{t_0 \leq t_j \leq t_k} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} W(t_k^+) \leq \prod_{t_0 \leq t_j < t_k} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} W(t_k) = U(t_k)$$

and for all $t \geq t_0$, we get

$$U(t_k^-) = \prod_{t_0 \leq t_j \leq t_{k-1}} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} W(t_k^-) \leq \prod_{t_0 \leq t_j < t_k} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} W(t_k) = U(t_k),$$

which implies that $U(t)$ is continuous on $[t_0, +\infty)$ and satisfies

$$\begin{aligned}
&U'(t) + \prod_{t_0 \leq t_k < t} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right) \frac{U^2(t)F(t)}{\varphi(t)} + \prod_{t_0 \leq t_k < t} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} G(t)\varphi(t) - \frac{\varphi'(t)U(t)}{\varphi(t)} \\
&= \prod_{t_0 \leq t_k < t} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} W'(t) + \prod_{t_0 \leq t_k < t} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right) \prod_{t_0 \leq t_k < t} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-2} \frac{F(t)}{\varphi(t)} W^2(t) \\
&\quad + \prod_{t_0 \leq t_k < t} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} G(t)\varphi(t) - \prod_{t_0 \leq t_k < t} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} \frac{\varphi'(t)}{\varphi(t)} W(t) \\
&= \prod_{t_0 \leq t_k < t} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} \left[W'(t) + W^2(t)\frac{F(t)}{\varphi(t)} - W(t)\frac{\varphi'(t)}{\varphi(t)} + G(t)\varphi(t)\right] \leq 0.
\end{aligned}$$

That is

$$U'(t) \leq -\prod_{t_0 \leq t_k < t} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right) \frac{F(t)}{\varphi(t)} U^2(t) + \frac{\varphi'(t)}{\varphi(t)} U(t) - \prod_{t_0 \leq t_k < t} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} G(t)\varphi(t).$$

Applying Lemma 2.6 with

$$X = \sqrt{\prod_{t_0 \leq t_k < t} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right) \frac{F(t)}{\varphi(t)}} U(t), \quad Y = \frac{\varphi'(t)}{2} \sqrt{\prod_{t_0 \leq t_k < t} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} \frac{1}{F(t)\varphi(t)}},$$

we have

$$\frac{\varphi'(t)}{\varphi(t)} U(t) - \prod_{t_0 \leq t_k < t} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right) \frac{F(t)}{\varphi(t)} U^2(t) \leq \frac{(\varphi'(t))^2}{4F(t)\varphi(t)} \prod_{t_0 \leq t_k < t} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1}.$$

Thus

$$U'(t) \le - \prod_{t_0 \le t_k < t} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}} \right)^{-1} \left[G(t)\varphi(t) - \frac{(\varphi'(t))^2}{4F(t)\varphi(t)} \right].$$

Integrating both sides from t_1 to t, we have

$$U(t) \le U(t_1) - \int_{t_1}^{t} \prod_{t_0 \le t_k < s} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}} \right)^{-1} \left[G(s)\varphi(s) - \frac{(\varphi'(s))^2}{4F(s)\varphi(s)} \right] ds.$$

Letting $t \to +\infty$, we have $\lim_{t \to +\infty} U(t) = -\infty$, which leads to a contradiction with $U(t) \ge 0$ and completes the proof. □

Theorem 3.3. *Assume that there exist functions $\varphi(t)$ and $\rho(s) \in C^1([0,+\infty),(0,+\infty))$ in such that $\varphi(t)$ is nondecreasing with respect to t, and the functions $H(t,s), h(t,s) \in C^1(D,\mathbb{R})$, in which $D = \{(t,s) | t \ge s \ge t_0 > 0\}$, such that*

(H_7) $H(t,t) = 0, \quad t \ge t_0; \quad H(t,s) > 0, \quad t > s \ge t_0,$

(H_8) $H_t'(t,s) \ge 0, \quad H_s'(t,s) \le 0,$

(H_9) $-\dfrac{\partial}{\partial s}[H(t,s)\rho(s)] - H(t,s)\rho(s)\dfrac{\varphi'(s)}{\varphi(s)} = h(t,s).$

If

$$\limsup_{t \to +\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \prod_{t_0 \le t_k < s} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}} \right)^{-1} \Pi(s) ds = +\infty, \tag{10}$$

where

$$\Pi(s) = G(s)\varphi(s)H(t,s)\rho(s) - \frac{1}{4} \frac{|h(t,s)|^2 \varphi(s)}{F(s)H(t,s)\rho(s)},$$

then every solution of the boundary value problem (E), (B_1) is oscillatory in G.

Proof. Assume that the boundary value problem (E), (B_1) has a nonoscillatory solution $u(x,t)$. Without loss of generality, assume that $u(x,t) > 0$, $(x,t) \in \Omega \times [0,+\infty)$. The case for $u(x,t) < 0$ can be considered in the same method. Proceeding as in the proof of Theorem 3.2, to get

$$U'(t) \le - \prod_{t_0 \le t_k < t} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}} \right) \frac{F(t)}{\varphi(t)} U^2(t) + \frac{\varphi'(t)}{\varphi(t)} U(t) - \prod_{t_0 \le t_k < t} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}} \right)^{-1} G(t)\varphi(t).$$

Multiplying the above inequality by $H(t,s)\rho(s)$ for $t \geq s \geq T$, and integrating from T to t, we have

$$\begin{aligned}\int_T^t U'(s)H(t,s)\rho(s)ds \leq & -\int_T^t \prod_{t_0\leq t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)\frac{F(s)}{\varphi(s)}U^2(s)H(t,s)\rho(s)ds \\ & + \int_T^t \frac{\varphi'(s)}{\varphi(s)}U(s)H(t,s)\rho(s)ds \\ & - \int_T^t \prod_{t_0\leq t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} G(s)\varphi(s)H(t,s)\rho(s)ds. \end{aligned} \tag{11}$$

Thus, we have

$$\begin{aligned}\int_T^t \prod_{t_0\leq t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} G(s)\varphi(s)H(t,s)\rho(s)ds \leq & \; U(T)H(t,T)\rho(T) \\ & + \int_T^t \left|h(t,s)U(s)\right| ds \\ & - \int_T^t \prod_{t_0\leq t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)\frac{F(s)}{\varphi(s)}U^2(s)H(t,s)\rho(s)ds. \end{aligned} \tag{12}$$

Applying Lemma 2.6 with

$$X = \sqrt{\prod_{t_0\leq t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)\frac{F(s)}{\varphi(s)}H(t,s)\rho(s)}U(s),$$

$$Y = \frac{1}{2}\left|h(t,s)\right|\sqrt{\prod_{t_0\leq t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1}\frac{\varphi(s)}{F(s)H(t,s)\rho(s)}},$$

we attain for $t > T \geq t_0$ that

$$\begin{aligned}\left|h(t,s)U(s)\right| - & \prod_{t_0\leq t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)\frac{F(s)}{\varphi(s)}H(t,s)\rho(s)U^2(s) \\ & \leq \frac{1}{4}\frac{\left|h(t,s)\right|^2\varphi(s)}{F(s)H(t,s)\rho(s)}\prod_{t_0\leq t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1}. \end{aligned} \tag{13}$$

In addition, from (12) and (13), we have

$$\int_T^t \prod_{t_0\le t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} G(s)\varphi(s)H(t,s)\rho(s)ds - \frac{1}{4}\int_T^t \frac{|h(t,s)|^2\varphi(s)}{F(s)H(t,s)\rho(s)}\prod_{t_0\le t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} ds$$
$$\le U(T)H(t,T)\rho(T) \le H(t,t_0)\rho(T)U(T), \quad t > T \ge t_0. \quad (14)$$

The rest of the proof is similar to the proof given by Philos[14]. □

Remark 3.4. In Theorem 3.3, by choosing $\rho(s) = \varphi(s) \equiv 1$, we have the following corollary.

Corollary 3.5. *Assume that the conditions of Theorem 3.3 hold, and*

$$\limsup_{t\to+\infty}\frac{1}{H(t,t_0)}\int_{t_0}^t \prod_{t_0\le t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1}\Gamma(s)ds = +\infty,$$

where

$$\Gamma(s) = G(s)H(t,s) - \frac{1}{4}\frac{|h(t,s)|^2}{F(s)H(t,s)},$$

then every solution of the boundary value problem $(E),(B_1)$ is oscillatory in G.

Remark 3.6. From Theorem 3.3 and Corollary 3.5, we can attain various oscillatory criteria by different choices of the weighted function $H(t,s)$. For example, choosing $H(t,s) = (t-s)^{n-1}$, $t \ge s \ge t_0$, in which $m > 2$ is an integer, then $h(t,s) = (n-1)(t-s)^{(n-3)/2}$, $t \ge s \ge t_0$. From Corollary 3.5, we have

Corollary 3.7. *If there exists an integer $m > 2$ such that*

$$\limsup_{t\to+\infty}\frac{1}{(t-t_0)^{n-1}}\int_{t_0}^t \prod_{t_0\le t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1}\left[G(s)(t-s)^{n-1} - \frac{1}{4}\frac{(n-1)^2}{(t-s)^2F(s)}\right]ds = +\infty, \quad (15)$$

then every solution of the boundary value problem (E), (B_1) is oscillatory in G.

Theorem 3.8. *Let the functions $H(t,s), h(t,s), \varphi(s)$ and $\rho(s)$ be as defined in Theorem 3.3. Additionally, suppose that*

$$0 < \inf_{s\ge t_0}\left\{\liminf_{t\to+\infty}\frac{H(t,s)}{H(t,t_0)}\right\} \le +\infty,$$

and

$$\limsup_{t\to+\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \prod_{t_0\le t_k<s} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} \frac{|h(t,s)|^2 \varphi(s)}{F(s)H(t,s)\rho(s)} ds < +\infty.$$

If there exists a function $A(t) \in C([t_0,+\infty),\mathbb{R})$ such that

$$\limsup_{t\to+\infty} \int_{t_0}^{t} \prod_{t_0\le t_k<s} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right) \frac{F(s)(A_+(s))^2}{\rho(s)\varphi(s)} ds = +\infty,$$

and for every $T \ge t_0$

$$\limsup_{t\to+\infty} \frac{1}{H(t,T)} \int_{T}^{t} \prod_{t_0\le t_k<s} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} \left[G(s)H(t,s)\varphi(s)\rho(s) - \frac{1}{4}\frac{|h(t,s)|^2 \varphi(s)}{F(s)H(t,s)\rho(s)}\right] ds \ge A(T),$$

where $A_+(s) = \max\{A(s),0\}$, then every solution of the boundary value problem $(E),(B_1)$ is oscillatory in G.

Proof. Assume that the boundary value problem $(E),(B_1)$ has a nonoscillatory solution $u(x,t)$. Without loss of generality, assume that $u(x,t) > 0$, $(x,t) \in \Omega\times[0,+\infty)$. The case for $u(x,t) < 0$ can be considered in the same method. Proceeding as in the proof of Theorem 3.3, we have (12) and (14). Then for $t > T \ge t_0$, we get

$$\limsup_{t\to+\infty} \frac{1}{H(t,T)} \int_{T}^{t} \prod_{t_0\le t_k<s} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} \left[G(s)H(t,s)\varphi(s)\rho(s) - \frac{1}{4}\frac{|h(t,s)|^2 \varphi(s)}{F(s)H(t,s)\rho(s)}\right] ds \le \rho(T)U(T).$$

The rest of the proof is similar to the proof in [23] and hence is omitted. □

Remark 3.9. In Theorem 3.8, by choosing $\rho(s) = \varphi(s) \equiv 1$, we get the following corollary.

Corollary 3.10. *Assume that the conditions of Theorem 3.8 hold, and assume that $\rho(s) = \varphi(s) \equiv 1$. If*

$$\limsup_{t\to+\infty} \frac{1}{H(t,T)} \int_{T}^{t} \prod_{t_0\le t_k<s} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} \left[G(s)H(t,s) - \frac{1}{4}\frac{|h(t,s)|^2}{F(s)H(t,s)}\right] ds \ge A(T),$$

for every $T \geq t_0$, where $A_+(s) = \max\{A(s), 0\}$, then every solution of the boundary value problem $(E), (B_1)$ is oscillatory in G.

Remark 3.11. Similar to Corollary 3.7, we can obtain the following corollary from Corollary 3.10.

Corollary 3.12. *Assume that the conditions of Theorem 3.8 hold, and*

$$\limsup_{t\to+\infty} \frac{1}{(t-t_0)^{n-1}} \int_{t_0}^{t} \prod_{t_0 \leq t_k < s} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}} \right)^{-1} \frac{(n-1)^2}{(t-s)^2 F(s)} ds < +\infty.$$

If there exists an integer $n > 2$ and function $A(t) \in C([0, +\infty), \mathbb{R})$ such that

$$\limsup_{t\to+\infty} \int_{t_0}^{t} \prod_{t_0 \leq t_k < s} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}} \right) F(s)(A_+(s))^2 ds = +\infty,$$

and for every $T \geq t_0$

$$\limsup_{t\to+\infty} \frac{1}{(t-t_0)^{n-1}} \int_{T}^{t} \prod_{t_0 \leq t_k < s} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}} \right)^{-1} \left[G(s)(t-s)^{n-1} - \frac{1}{4} \frac{(n-1)^2}{(t-s)^2 F(s)} \right] ds \geq A(T),$$

where $A_+(s) = \max\{A(s), 0\}$, then every solution of the boundary value problem $(E), (B_1)$ is oscillatory in G.

4. Oscillation of the Problem (E) and (B_2)

In this section, we establish sufficient conditions for the oscillation of all solutions of the problem (E), (B_2).

Lemma 4.1. *If the functional impulsive differential inequality*

$$\left.\begin{aligned} &\tilde{Z}^{(m)}(t) + G(t)\tilde{Z}(\theta(t)) \leq 0, \quad t \neq t_k \\ &a_k^{(i)} \leq \frac{\dfrac{\partial^{(i)} \tilde{Z}(t_k^+)}{\partial t^{(i)}}}{\dfrac{\partial^{(i)} \tilde{Z}(t_k)}{\partial t^{(i)}}} \leq b_k^{(i)}, \quad k = 1, 2, \cdots, \quad i = 0, 1, 2, \cdots, m-1 \end{aligned}\right\} \tag{16}$$

has no eventually positive solution, then every solution of the boundary value problem defined by (E) and (B_2) is oscillatory in G.

Proof. Assume that there exist a nonoscillatory solution $u(x,t)$ of the boundary value problem (E), (B_2) and $u(x,t)>0$. By the hypothesis (H_1) and (H_3), that there exists a $t_1>t_0>0$ such that $\tau(t)\geq t_0$, $\sigma(t,\xi),\rho(t,\xi)\geq t_0$ for $(t,\xi)\in[t_1,+\infty)\times[a,b]$ for $t\geq t_1$, then

$$\begin{aligned}
&u(x,\tau(t))>0 \quad \text{for} \quad (x,t)\in\Omega\times[t_1,+\infty),\\
&u(x,\sigma(t,\xi))>0 \quad \text{for} \quad (x,t,\xi)\in\Omega\times[t_1,+\infty)\times[a,b]\\
\text{and}\quad &u(x,\rho(t,\xi))>0 \quad \text{for} \quad (x,t,\xi)\in\Omega\times[t_1,+\infty)\times[a,b].
\end{aligned}$$

For $t\geq t_0$, $t\neq t_k$, $k=1,2,\cdots$, multiplying both sides of equation (E) by $1/|\Omega|$ and integrating with respect to x over the domain Ω, we obtain

$$\left.\begin{aligned}
&\frac{d^m}{dt^m}\left[\frac{1}{|\Omega|}\int_\Omega u(x,t)dx+\frac{1}{|\Omega|}\int_\Omega c(t)u(x,\tau(t))dx\right]\\
&+\frac{1}{|\Omega|}\int_\Omega\int_a^b q(x,t,\xi)f(u(x,\sigma(t,\xi)))d\eta(\xi)dx\\
&=a(t)\frac{1}{|\Omega|}\int_\Omega\Delta u(x,t)dx-\frac{1}{|\Omega|}\int_\Omega\int_a^b b(t,\xi)\Delta u(x,\rho(t,\xi))d\eta(\xi)dx.
\end{aligned}\right\}\tag{17}$$

By Green's formula and boundary condition (B_2),

$$\int_\Omega\Delta u(x,t)dx=\int_{\partial\Omega}\frac{\partial u}{\partial\gamma}dS=-\int_{\partial\Omega}\mu(x,t)u(x,t)dS\leq 0,\tag{18}$$

and

$$\int_\Omega\Delta u(x,\rho(t,\xi))dx=\int_{\partial\Omega}\frac{\partial u(x,\rho(t,\xi))}{\partial\gamma}dS=-\int_{\partial\Omega}\mu(x,\rho(t,\xi))u(x,\rho(t,\xi))dS\leq 0\tag{19}$$

where dS is surface element on $\partial\Omega$. Also from (H_2) and Jensen's inequality, we have

$$\begin{aligned}
&\frac{1}{|\Omega|}\int_\Omega\int_a^b q(x,t,\xi)f(u(x,\sigma(t,\xi)))d\eta(\xi)dx\\
&\qquad\geq\int_a^b Q(t,\xi)\frac{1}{|\Omega|}\int_\Omega f(u(x,\sigma(t,\xi)))dxd\eta(\xi)\\
&\qquad=\int_a^b Q(t,\xi)\,\epsilon\frac{1}{|\Omega|}\int_\Omega u(x,\sigma(t,\xi))dxd\eta(\xi)\\
&\qquad\geq\epsilon\int_a^b Q(t,\xi)\tilde{V}(\sigma(t,\xi))d\eta(\xi).
\end{aligned}\tag{20}$$

In view of (17)-(20), yield

$$\frac{d^m}{dt^m}\left[\tilde{V}(t)+c(t)\tilde{V}(\tau(t))\right]+\epsilon\int_a^b Q(t,\xi)\tilde{V}(\sigma(t,\xi))d\eta(\xi)\leq 0. \tag{21}$$

Set $\tilde{Z}(t)=\tilde{V}(t)+c(t)\tilde{V}(\tau(t))$. Equation (21), can be written as

$$Z^{(m)}(t)+\epsilon\int_a^b Q(t,\xi)\tilde{V}(\sigma(t,\xi))d\eta(\xi)\leq 0, \quad t\neq t_k. \tag{22}$$

Rest of the proof is parallel to the Lemma 3.1, and hence the details are omitted.

□

Theorem 4.2. *If there exists a function $\tilde{\varphi}(t)\in C^1([0,+\infty),(0,+\infty))$ which is nondecreasing with respect to t, such that*

$$\int_{t_1}^{+\infty}\prod_{t_0\leq t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1}\left[\tilde{\varphi}(s)G(s)-\frac{(\tilde{\varphi}'(s))^2}{4F(s)\tilde{\varphi}(s)}\right]ds=\infty,$$

then every solution of the boundary value problem (E), (B_2) is oscillatory in G.

Theorem 4.3. *Assume that there exist functions $\tilde{\varphi}(t)$ and $\tilde{\rho}(s)\in C^1([0,+\infty),(0,+\infty))$ such that $\tilde{\varphi}(t)$ is nondecreasing. Assume that the functions there exist two functions $H(t,s)$, $h(t,s)\in C^1(D,\mathbb{R})$, in which $D=\{(t,s)|t\geq s\geq t_0>0\}$, such that $(H_7)-(H_9)$ hold. If*

$$\limsup_{t\to+\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\prod_{t_0\leq t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1}\tilde{\Pi}(s)ds=\infty,$$

where

$$\tilde{\Pi}(s)=G(s)\tilde{\varphi}(s)H(t,s)\tilde{\rho}(s)-\frac{1}{4}\frac{|h(t,s)|^2\tilde{\varphi}(s)}{F(s)H(t,s)\tilde{\rho}(s)},$$

then every solution of the boundary value problem (E), (B_2) is oscillatory in G.

Remark 4.4. In Theorem 4.3, by choosing $\tilde{\rho}(s)=\tilde{\varphi}(s)\equiv 1$, we have the following corollary.

Corollary 4.5. *Assume that the conditions $(H_7)-(H_9)$ hold, and*

$$\limsup_{t\to+\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\prod_{t_0\leq t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1}\Gamma(s)ds=\infty,$$

then every solution of the boundary value problem (E), (B_2) is oscillatory in G.

Remark 4.6. From Theorem 4.3 and Corollary 4.5, we can attain various oscillatory criteria by different choices of the weighted function $H(t,s)$. For example, choosing $H(t,s) = (t-s)^{n-1}$, $t \geq s \geq t_0$, in which $n > 2$ is an integer, then $h(t,s) = (n-1)(t-s)^{(n-3)/2}$, $t \geq s \geq t_0$. From Corollary 4.5, we get

Corollary 4.7. *If there exists an integer $n > 2$ such that*

$$\limsup_{t\to+\infty} \frac{1}{(t-t_0)^{n-1}} \int_{t_0}^{t} \prod_{t_0 \leq t_k < s} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}} \right)^{-1} \left[G(s)(t-s)^{n-1} - \frac{1}{4} \frac{(n-1)^2}{(t-s)^2 F(s)} \right] ds = +\infty,$$

then every solution of the boundary value problem (E), (B_2) is oscillatory in G.

Theorem 4.8. *Let the functions $H(t,s)$, $h(t,s)$, $\tilde{\varphi}(s)$ and $\tilde{\rho}(s)$ be as defined in Theorem 4.3. Additionally, suppose that*

$$0 < \inf_{s \geq t_0} \left\{ \liminf_{t\to+\infty} \frac{H(t,s)}{H(t,t_0)} \right\} \leq +\infty,$$

and

$$\limsup_{t\to+\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \prod_{t_0 \leq t_k < s} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}} \right)^{-1} \frac{|h(t,s)|^2 \tilde{\varphi}(s)}{F(s)H(t,s)\tilde{\rho}(s)} ds < +\infty.$$

If there exists a function $\tilde{A}(t) \in C([t_0,+\infty),\mathbb{R})$ such that

$$\limsup_{t\to+\infty} \int_{t_0}^{t} \prod_{t_0 \leq t_k < s} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}} \right) \frac{F(s)(\tilde{A}_+(s))^2}{\tilde{\rho}(s)\tilde{\varphi}(s)} ds = +\infty,$$

and for every $T \geq t_0$

$$\limsup_{t\to+\infty} \frac{1}{H(t,T)} \int_{T}^{t} \prod_{t_0 \leq t_k < s} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}} \right)^{-1} \left[G(s)H(t,s)\tilde{\varphi}(s)\tilde{\rho}(s) - \frac{1}{4} \frac{|h(t,s)|^2 \tilde{\varphi}(s)}{F(s)H(t,s)\tilde{\rho}(s)} \right] ds$$
$$\geq \tilde{A}(T),$$

where $\tilde{A}_+(s) = \max\{\tilde{A}(s), 0\}$, then every solution of the boundary value problem $(E), (B_2)$ is oscillatory in G.

Remark 4.9. In Theorem 4.8, by choosing $\tilde{\rho}(s) = \tilde{\varphi}(s) \equiv 1$, we attain the following corollary.

Corollary 4.10. *Assume that the conditions of Theorem 4.8 hold and assume that* $\tilde{\rho}(s) = \tilde{\varphi}(s) \equiv 1$. *If*

$$\limsup_{t\to+\infty} \frac{1}{H(t,T)} \int_T^t \prod_{t_0 \le t_k < s} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}} \right)^{-1} \left[G(s)H(t,s) - \frac{1}{4} \frac{|h(t,s)|^2}{F(s)H(t,s)} \right] ds \ge \tilde{A}(T),$$

for every $T \ge t_0$, *then every solution of the boundary value problem* (E), (B_2) *is oscillatory in* G.

Remark 4.11. Similar to Corollary 4.7, we can obtain the following corollary from Corollary 4.10.

Corollary 4.12. *Assume that the conditions of Theorem 4.8 hold, and*

$$\limsup_{t\to+\infty} \frac{1}{(t-t_0)^{n-1}} \int_{t_0}^t \prod_{t_0 \le t_k < s} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}} \right)^{-1} \frac{(n-1)^2}{(t-s)^2 F(s)} ds < \infty.$$

If there exists an integer $n > 2$ *and function* $\tilde{A}(t) \in C([0,+\infty),\mathbb{R})$ *such that*

$$\limsup_{t\to+\infty} \int_{t_0}^t \prod_{t_0 \le t_k < s} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}} \right) F(s)(\tilde{A}_+(s))^2 ds = \infty,$$

and for every $T \ge t_0$

$$\limsup_{t\to+\infty} \frac{1}{(t-t_0)^{n-1}} \int_T^t \prod_{t_0 \le t_k < s} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}} \right)^{-1} \left[G(s)(t-s)^{n-1} - \frac{1}{4} \frac{(n-1)^2}{(t-s)^2 F(s)} \right] ds \ge \tilde{A}(T),$$

then every solution of the boundary value problem $(E), (B_2)$ *is oscillatory in* G.

5. Examples

In this part, we present couple of examples to point up our results established in Section 3 and Section 4.

Example 5.1. *Consider the following equation*

$$\frac{\partial^6}{\partial t^6}\left(u(x,t) + \frac{1}{5}u(x,t-\pi) \right) + \frac{6}{5}\int_{-\pi/2}^{-\pi/4} u(x,t+2\xi)d\xi \quad = \frac{4}{5}\Delta u(x,t) - \frac{6}{5}\int_{-\pi/2}^{-\pi/4} \Delta u(x,t+2\xi)d\xi, \quad t \tag{23}$$

for $(x,t) \in (0,\pi) \times [0,+\infty)$*, with the boundary condition*

$$u(0,t) = u(\pi,t) = 0, \quad t \neq t_k. \tag{24}$$

Here $\Omega = (0,\pi)$, $m = 6$, $a_k^{(0)} = b_k^{(0)} = \frac{k+1}{k}$, $a_k^{(i)} = b_k^{(i)} = 1$, $i = 1,2,3,4,5$, $c(t) = \frac{1}{5}$, $\tau(t) = t - \pi$, $Q(t,\xi) = \frac{6}{5}$, $f(u) = u$, $\sigma(t,\xi) = \rho(t,\xi) = t + 2\xi$, $a(t) = \frac{4}{5}$, $b(t,\xi) = \frac{6}{5}$, $\eta(\xi) = \xi$, $\theta(t) = t$, $\theta'(t) = 1$, $\epsilon = 1$*. Since* $t_0 = 1$, $t_k = 2^k$, $g_0 = \frac{4}{5}$, $G(s) = \frac{6\pi}{25}$, $F(s) = s^4$*. Then from the hypotheses* $(H_1) - (H_6)$ *hold, moreover*

$$\begin{aligned}
\lim_{t\to+\infty} \int_{t_0}^{t} \prod_{t_0 \le t_k < s} \frac{a_k^{(0)}}{b_k^{(i)}} ds &= \int_1^{+\infty} \prod_{1<t_k<s} \frac{k}{k+1} ds \\
&= \int_1^{t_1} \prod_{1<t_k<s} \frac{k}{k+1} ds + \int_{t_1^+}^{t_2} \prod_{1<t_k<s} \frac{k}{k+1} ds + \int_{t_2^+}^{t_3} \prod_{1<t_k<s} \frac{k}{k+1} ds + \cdots \\
&= 1 + \frac{1}{2} \times 2 + \frac{1}{2} \times \frac{2}{3} \times 2^2 + \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times 2^3 + \cdots \\
&= \sum_{n=0}^{+\infty} \frac{2^n}{n+1} = +\infty.
\end{aligned}$$

Thus, the condition (3.15) *reads,*

$$\limsup_{t\to+\infty} \frac{1}{(t-1)^5} \left\{ \int_1^t \prod_{1<t_k<s} \frac{k}{k+1} \left[\frac{6\pi}{25}(t-s)^5 - \frac{25}{4s^4(t-s)^2} \right] ds \right\} = +\infty.$$

Therefore all the conditions of the Corollary 3.7 are satisfied. Therefore, every solution of equation (23)-(24) *is oscillatory in* G*. In fact* $u(x,t) = \sin x \cos t$ *is such a solution.*

Example 5.2. *Consider the following equation of the form*

$$\left.\begin{aligned}
&\frac{\partial^4}{\partial t^4}\left(u(x,t)+\frac{1}{2(t+1)}u(x,t-3\pi)\right)+\frac{1}{2}\int_{-\pi}^{0}u(x,t+\xi)d\xi\\
&\quad=\left(\frac{12}{(t+1)^5}-\frac{6}{(t+1)^3}+\frac{1}{2(t+1)}-1\right)\Delta u(x,t)\\
&\quad+\left(\frac{1}{2}\left(1-\frac{12}{(t+1)^4}+\frac{2}{(t+1)^2}\right)\right)\int_{-\pi}^{0}\Delta u(x,t+\xi)d\xi, \quad t>1,\ t\neq t_k,\\
&u(x,(t_k)^+)=\frac{k+1}{k}u(x,t_k),\\
&\frac{\partial^{(i)}}{\partial t^{(i)}}u(x,(t_k)^+)=\frac{\partial^{(i)}}{\partial t^{(i)}}u(x,t_k), \quad i=1,2,3, \quad k=1,2,\cdots
\end{aligned}\right\} \tag{25}$$

for $(x,t)\in(0,\pi)\times[0,+\infty)$, *with the boundary condition*

$$u_x(0,t)=u_x(\pi,t)=0, \quad t\neq t_k. \tag{26}$$

Here $\Omega=(0,\pi)$, $m=4$, $\mu(x,t)=1$, $a_k^{(0)}=b_k^{(0)}=\dfrac{k+1}{k}$, $a_k^{(i)}=b_k^{(i)}=1$, $i=1,2,3$, $c(t)=\dfrac{1}{2(t+1)}$, $\tau(t)=t-3\pi$, $Q(t,\xi)=\dfrac{1}{2}$, $f(u)=u$, $\sigma(t,\xi)=\rho(t,\xi)=t+\xi$, $a(t)=\dfrac{12}{(t+1)^5}-\dfrac{6}{(t+1)^3}+\dfrac{1}{2(t+1)}-1$, $b(t,\xi)=\dfrac{1}{2}\left(1-\dfrac{12}{(t+1)^4}+\dfrac{2}{(t+1)^2}\right)$, $\eta(\xi)=\xi$, $\theta(t)=t^2$, $\theta'(t)=2t$, $\epsilon=1$. *Since* $t_0=1$, $t_k=2^k$, $g_0=1-\dfrac{1}{2(t+\xi+1)}$, $G(s)=\dfrac{\pi}{2}\left(1-\dfrac{1}{2(t+\xi+1)}\right)$, $F(s)=2s^5$. *Then hypotheses* $(H_1)-(H_6)$ *hold. Thus,*

$$\limsup_{t\to+\infty}\frac{1}{(t-1)^3}\left\{\int_1^t\prod_{1<t_k<s}\frac{k}{k+1}\left[\frac{\pi}{2}\left(1-\frac{1}{2(s+\xi+1)}\right)(t-s)^3-\frac{9}{8s^5(t-s)^2}\right]ds\right\}=+\infty.$$

Therefore all the conditions of the Corollary 4.7 are satisfied. Therefore, every solution of equation (25)-(26) *is oscillatory in* G. *In fact* $u(x,t)=\cos x\sin t$ *is such a solution.*

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