Original Research Article

J. Comp. Matha. 01 (2017) 2456–8686

SACRED HEART RESEARCH PUBLICATIONS

Journal of Computational Mathematica

Journal homepage: www.shcpub.edu.in

ISSN: 2456-8686

Restrained Domination Number of Some Path Related Graphs

[1]S. K. Vaidya, and [2]P. D. Ajani,

Received on 20 Jan 2017, Accepted on 08 April 2017

Abstract. A dominating set $S \subseteq V(G)$ of a graph G is called restrained dominating set if every vertex in $V(G) - S$ is adjacent to a vertex in S and to a vertex in $V(G) - S$. The restrained domination number of G, denoted by $\gamma_r(G)$, is the minimum cardinality of a restrained dominating set of G.

Key words: Dominating set, restrained dominating set, restrained domination number

AMS Subject classification: 05C69, 05C38

1. Introduction

By a graph G, we mean a finite, connected and undirected graph without loops or multiple edges with vertex set $V(G)$ and edge set $E(G)$. For any vertex $v \in V(G)$ the open neighborhood of v is $N(v) = \{u \in V(G)/uv \in E(G)\}$ and the closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. For any graph theoretic terminology and notation we refer to Harary [6] and the terms related to domination are used in the sense of Haynes *et al.* [8].

The concept of domination in graph is one of the most emerging concepts within and outside of graph theory. A set $S \subseteq V(G)$ in a graph G is a dominating set if every vertex of $V(G) - S$ is adjacent to at least one vertex in S. The minimum cardinality of dominating set S is called domination number denoted by $\gamma(G)$. A vertex in a graph G dominates itself and its neighbors. A set $S \subseteq V(G)$ is a restrained dominating set if every vertex in $V(G) - S$ is adjacent to a vertex in S

[1]Corresponding Author: E-mail: samirkvaidya@yahoo.co.in [2]e-mail: paragajani@gmail.com

[1]Department of Mathematics, Saurashtra University, Rajkot, Gujarat, India - 360005

[2]Shree M. & N. Virani Science College (Autonomous), Rajkot, Gujarat, India - 360005

and to a vertex in $V(G) - S$. The restrained domination number of G, denoted by $\gamma_r(G)$, is the minimum cardinality of a restrained dominating set of G. There are various domination models available in the literature. Some of them are total domination [3], equitable domination [11], global domination [13], steiner domination [14], independent domination [1] are among worth to mention. The concept of restrained domination was conceived by Telle and Proskurowski [12] as a vertex partitioning problem. Some variants of restrained domination are restrained 2-domination [7], restrained roman domination [9] and trees with equal domination [4]. The present paper is aimed to explore the concept of restrained domination in graph. We investigate the restrained domination number of some path related graphs.

Proposition 1.1. [5] *If $n \geqslant 1$ is an integer, then* $\gamma_r(P_n) = n - 2\left\lfloor \dfrac{n-1}{3} \right\rfloor$.

Proposition 1.2. [2] $\gamma(P_n^2) = \left\lceil \dfrac{n}{5} \right\rceil$, *for* $n \geqslant 3$.

Proposition 1.3. [10] $\gamma(T(P_n)) = \left\lceil \dfrac{2n-1}{5} \right\rceil$, *for* $n \geqslant 2$.

2. Main Results

Definition 2.1. Let G be a graph with $V(G) = S_1 \cup S_2 \cup S_3 \cup ... \cup S_t \cup T$, where each S_i is set of all the vertices of same degree with at least two elements and $T = V(G) - \bigcup\limits_{i=1}^{t} S_i$. The degree splitting of G, denoted by $DS(G)$, is obtained from G by adding vertices $w_1, w_2, w_3, ..., w_t$ and joining w_i to each vertex of S_i for $1 \leqslant i \leqslant t$. Note that if $V(G) = \bigcup\limits_{i=1}^{t} S_i$ then $T = \phi$.

We derive the following result for the degree splitting graph of P_n.

Theorem 2.2. $\gamma_r(DS(P_n)) = 2$; for $n \geqslant 4$.

Proof. Let P_n be a path of order n with vertex set $V(P_n) = \{v_1, v_2, ..., v_n\}$.

In order to obtain $DS(P_n)$ from $P_n (n \geqslant 4)$, we add two vertices u_1 and u_2 corresponding to $S_1 = \{v_1, v_n\}$ and $S_2 = \{v_2, v_3, \ldots, v_{n-1}\}$ respectively, join u_1 to each vertex of S_1 and u_2 to each vertex of S_2. Then $|V(DS(P_n))| = n+2$ and there does not exist a vertex $v \in V(DS(P_n))$ such that $d(v) = n+1$. Therefore $\gamma(DS(P_n)) > 1$ and $\gamma_r(DS(P_n)) > 1$.

If $S \subseteq V(DS(P_n))$ is a dominating set then $|S| \geqslant 2$. As $N(u_1) = \{v_1, v_n\}$ and $N(u_2) = \{v_2, v_3, v_4, ..., v_{n-1}\}$ implies $S = \{u_1, u_2\}$ is the only required dominating set with minimum cardinality, consequently $N[S] = V(DS(P_n))$ with $\{u_1, u_2\} \in S$ and $\{v_1, v_2, v_3, ..., v_n\} \in V - S$. Also every vertex of $V - S$ is adjacent to at least one other vertex of $V - S$.

That is, S is a required restrained dominating set with $|S| = 2$. Hence $\gamma_r(DS(P_n)) = 2$; for $n \geqslant 3$. □

Definition 2.3. The switching of a vertex v of G means removing all the edges incident to v and adding edges joining v to every vertex which is not adjacent to v in G. We denote the resultant graph by $\widetilde{G}$.

We derive the following result for switching of a pendant vertex in P_n.

Theorem 2.4. $\gamma_r(\widetilde{P_n}) = \begin{cases} 3; & \text{for } n = 3 \\ 2; & \text{for } n \geqslant 2 - \{3\} \end{cases}$.

Proof. Let P_n be a path of order n with vertex set $V(P_n) = \{v_1, v_2, ..., v_n\}$ and edge set $E(P_n)$, where $d(v_1) = 1 = d(v_n)$ and $d(v_k) = 2$; for $2 \leqslant k \leqslant n-1$.

Let $\widetilde{P_n}$ be the graph obtained by switching either of the pendant vertices of path P_n with vertex set $V(\widetilde{P_n}) = V(P_n)$. We prove the result by considering following cases.

Case-1: For $n = 2$

Let $V(P_2) = \{v_1, v_2\}$ and $d(v_1) = 1 = d(v_2)$. By switching either of the pendant vertices in P_2, we obtain two isolated vertices v_1 and v_2 so they dominate themselves.

If $S \subseteq V(\widetilde{P_2})$ is a dominating set then $\{v_1, v_2\} \in S$. As $S = V(P_2)$ and $V - S = \phi$ implies that $N[S] = V(P_2)$. That is, $S = \{v_1, v_2\}$ is restrained dominating set with minimum cardinality.

Hence $\gamma_r(\widetilde{P_2}) = 2$.

Case-2: For $n = 3$

Let $V(P_3) = \{v_1, v_2, v_3\}$ and $d(v_1) = 1 = d(v_3)$. By switching either of the pendant vertices in P_3, we obtain $(\widetilde{P_3}) = P_3$. But according to Proposition 1.1 $\gamma_r(\widetilde{P_3}) = \gamma_r(P_3) = 3$

Hence $\gamma_r(\widetilde{P_3}) = 3$.

Case-3: For $n \geqslant 4$

Let $V(P_n) = \{v_1, v_2, v_3, ..., v_n\}$, with $d(v_1) = 1 = d(v_n)$ and $\widetilde{P_n}$ be the graph obtained by switching of v_1 or v_n. Without loss of generality we switch the vertex v_1. Since $|V(\widetilde{P_n})| = n$ and there does not exist a vertex v such that $d(v) = n - 1$ implies that $\gamma(\widetilde{P_n}) > 1$.

In $\widetilde{P_n}$, $d(v_1) = n - 2$, v_1 dominates every element of $\{v_3, v_4, ..., v_n\}$ and v_2 dominates itself. Therefore v_1 and v_2 are enough to dominate all the vertices of $V(\widetilde{P_n})$. Thus $S = \{v_1, v_2\}$ is a dominating set with minimum cardinality. Also $N[S] = V(\widetilde{P_n})$ as $\{v_3, v_4, ..., v_n\} \in V - S$. Thus every vertex of $V - S$ is adjacent to at least one other vertex of $V - S$. Therefore S is a restrained dominating set with $|S| = 2$.

Hence $\gamma(\widetilde{P_n}) = 2$ for $n \geqslant 4$. □

Definition 2.5. The square of a graph G, denoted by G^2, has the same vertex set as of G and two vertices are adjacent in G^2 if they are at distance 1 or 2 apart in G.

We derive the following result for P_n^2.

Theorem 2.6. $\gamma_r(P_n^2) = \left\lceil \frac{n}{5} \right\rceil$ $n \geqslant 3$.

Proof. Let $v_1, v_2, ..., v_n$ be the vertices of path P_n. Then $|V(P_n^2)| = n$.

We divide the proof in following two cases.

Case-1: $n \equiv 0$ *or* 3 *or* $4 \pmod 5$

If $S \subseteq V(P_n^2)$ is a dominating set then $S = \{u_i \in V(P_n^2);\ i \equiv 0 \text{ or } 3 \text{ or } 4 (\bmod 5)\}$ and according to Proposition 1.2, $|S| = \left\lceil \frac{n}{5} \right\rceil$. Also every vertex of $V - S$ is adjacent to at least one other vertex of $V - S$. Therefore $|S| = \left\lceil \frac{n}{5} \right\rceil$ is the only restrained dominating set with minimum cardinality.

Hence $\gamma_r(P_n^2) = \left\lceil \frac{n}{5} \right\rceil$, for $n \equiv 0$ *or* 3 *or* $4 \pmod 5$.

Case-2: $n \equiv 1$ *or* $2 (\bmod 5)$

If $S \subseteq V(P_n^2)$ is a dominating set then $S = S' \cup \{v_n\}$ where $S' = \{u_i \in V(P_n^2);\ i \equiv 1 \text{ or } 2 (\bmod 5)\}$ and according to Proposition 1.2, $|S| = \left\lceil \frac{n}{5} \right\rceil$ also every vertex of $V - S$ is adjacent to at least one other vertex of $V - S$. Therefore $|S| = \left\lceil \frac{n}{5} \right\rceil$ is the only restrained dominating set with minimum cardinality.

Hence $\gamma_r(P_n^2) = \left\lceil \frac{n}{5} \right\rceil$, for $n \equiv 1$ *or* $2 \pmod 5$.

$\square$

Definition 2.7. Let G be a graph with two or more vertices then the total graph $T(G)$ of graph G is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices are adjacent whenever they are either adjacent or incident in G.

We derive the following result for total graph of path.

Theorem 2.8. $\gamma_r(T(P_n)) = \left\lceil \frac{2n-1}{5} \right\rceil$, *for* $n \geqslant 2$.

Proof. Let P_n be a path of order n with vertex set $V(P_n) = \{v_1, v_2, ..., v_n\}$ and edge set $E(P_n)$, where $d(v_1) = 1 = d(v_n)$ and $d(v_k) = 2; 1 < k < n$. Let $T(P_n)$ be a total graph of path P_n whose vertex set is $V(T(P_n)) = V(P_n) \cup E(P_n)$ and two vertices are adjacent whenever they are either adjacent or incident in G, so $|V(T(P_n))| = 2n - 1$ where $d(v_1) = 2 = d(v_n)$ and $d(v_k) = 4$, for $1 < k < n$.

We divide this proof into two cases as follows.

Case-1: $n \equiv 0 \ or \ 3 (\bmod 5)$.

If $S \subseteq V(T(P_n))$ is a dominating set then $S = \{v_i \in V(T(P_n));$ $i \equiv 0 \ or \ 3 (\bmod 5)\}$ and according to Proposition 1.3, $|S| = \left\lceil \frac{2n-1}{5} \right\rceil$. Also every vertex of $V - S$ is adjacent to at least one other vertex of $V - S$, so that $|S| = \left\lceil \frac{2n-1}{5} \right\rceil$ is the only restrained dominating set with minimum cardinality.

Hence $\gamma_r(T(P_n)) = \left\lceil \frac{2n-1}{5} \right\rceil$, for $n \equiv 0 \ or \ 3 (\bmod 5)$.

Case-2: $n \equiv 1 \ or \ 2 \ or \ 4 (\bmod 5)$.

If $S \subseteq V(T(P_n))$ is a dominating set then $S = S' \cup \{v_n\}$, where $S' = \{v_i \in V(T(P_n));\ i \equiv 1 \ or \ 2 \ or \ 4 (\bmod 5)\}$ and according to Proposition 1.3, $|S| = \left\lceil \frac{2n-1}{5} \right\rceil$. Also every vertex of $V - S$ is adjacent to at least one other vertex of $V - S$, so that $|S| = \left\lceil \frac{2n-1}{5} \right\rceil$ is the only restrained dominating set with minimum cardinality.

Hence $\gamma_r(T(P_n)) = \left\lceil \frac{2n-1}{5} \right\rceil$, for $n \equiv 1$ *or* 2 *or* $4 (\mathrm{mod}\, 5)$. $\square$

3. Concluding Remarks

There are many variants of dominating sets available in the literature and study of related parameters is also interesting. The restrained dominating set relates the vertex set and its complements. We have obtained restrained domination number of various graphs by means of graph operations on P_n. Similar results can be proved in the context of various domination models and different graph operations.

References

[1] S. Ao, E. J. Cockayne, G. MacGillivray and C. M. Mynhardt, Domination Critical Graphs with Higher Independent Domination Numbers, J. Graph Theory, (22)(1996), 9-14.

[2] G. Chartrand and P. Zhang, *Introduction to Graph Theory*, McGraw-Hill, Boston, 2005.

[3] E. J. Cockayne, R. M. Dawes and S. T. Hedetniemi, Total Domination in Graphs, Networks, (10)(1980), 211-219.

[4] P. Dankelmann, J. H. Hattingh, M. A. Henning and H. C. Swart, Trees with Equal Domination and Restrained Domination Numbers, J. Global Optim. (34)(2006), 597-607.

[5] G. S. Domke, J. H. Hattingh, S. T. Hedetiniemi, R. C. Laskar and L. R. Markus, Restrained domination in graphs, Discrete Mathematics, (203)(1999), 61-69.

[6] F. Harary, *Graph Theory*, Addison-Wesley, Reading, Mass, 1972.

[7] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

[8] J. J. Mohan and I. Kelkar, Restrained 2-Domination Number of Complete Grid Graph, International Journal of Applied Mathematics and Computation, 4(4)(2012), 352-358.

[9] P. R. L. Pushpam and S. Padmapriea, Restrained Roman Domination in Graphs, Transactions on Combinatorics, 4(1)(2015), 1-17.

[10] M. S. Shukla, *Coloring and Domination Problems in Graphs-Some New Perspective*, Ph.D. Thesis, Saurashtra University, Rajkot(India), 2016.

[11] V. Swaminathan and K. M. Dharmalingam, Degree Equitable Dominations on Graphs, Kragujevac Journal of Mathematics, 35(1)(2011), 191-197.

[12] J. A. Telle and A. Proskurowski, Algorithms for vertex partitioning problems on partial k-trees, SIAM J. Discrete Mathematics, (10)(1997), 529-550.

[13] S. K. Vaidya and R. M. Pandit, Some New Results on Global Dominating Sets, ISRN Discrete Mathematics, (2012), 1-6.

[14] S. K. Vaidya and R. N. Mehta, Steiner Domination Number of Some Wheel Related Graphs, International Journal of Mathematics and Soft Computing, 5(2)(2015), 15-19.