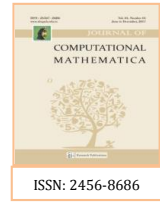




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Fibonacci Sequence and its Sum by Second Order Difference Operator with Polynomial Factorial

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ABSTRACT. In this paper, we introduce second order difference operator with polynomial factorial and its inverse by which we obtain advanced Fibonacci sequence and its sum. Some theorems and interesting results on the sum of the terms of second order Fibonacci sequence are derived. Suitable examples are provided to illustrate our results and verified by MATLAB.

Key words: Difference operator, polynomial factorial, Fibonacci sequence, Closed form solution, Fibonacci summation.

AMS Subject classification: 39A70, 39A10, 47B39, 65J10, 65Q10.

1. INTRODUCTION

In 1984, Jerzy Popenda [5] introduced a particular type of difference operator on $u(k)$ as $\Delta_\alpha u(k) = u(k+1) - \alpha u(k)$. In 1989, Miller and Rose [8] introduced the discrete analogue of the Riemann-Liouville fractional derivative and its inverse $\Delta_h^{-\nu} f(t)$ ([1, 4]). The sum of m^{th} partial sums of products of higher powers of arithmetic and geometric progressions are derived in [9] by replacing h by ℓ , ν by m and $f(t)$ by $u(k)$ in $\Delta_h^{-\nu} f(t)$.

In 2011, M.Maria Susai Manuel, et.al, [7] extended the operator Δ_α to generalized α -difference operator as $\Delta_{\alpha(\ell)} v(k) = v(k+\ell) - \alpha v(k)$ for the real valued function $v(k)$. In 2014, G.Britto Antony Xavier, et.al, [2] introduced

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q -difference operator as $\Delta_q v(k) = v(qk) - v(k)$, $q \in (0, \infty)$ and obtained finite series solution to the corresponding generalized q -difference equation $\Delta_q v(k) = u(k)$. In this paper, we introduce second order difference operator by which we obtain second order polynomial factorial Fibonacci sequence and its sum. suitable examples verified by MATLAB are provided.

2. BASIC DEFINITIONS AND EXAMPLES

Fibonacci and Lucas numbers cover a wide range of interest in modern mathematics as they appear in the comprehensive works of Koshy [6] and Vajda [10]. The k -Fibonacci sequence introduced by Falcon and Plaza [3] depends only on one integer parameter k and is defined as

$$F_{k,0} = 0, \quad F_{k,1} = 1 \quad \text{and} \quad F_{k,n+1} = kF_{k,n} + F_{k,n-1}, \quad \text{where } n \geq 1, k \geq 1.$$

In particular, if $k = 2$, the Pell sequence is obtained as

$$P_0 = 0, \quad P_1 = 1 \quad \text{and} \quad P_{n+1} = 2P_n + P_{n-1} \quad \text{for } n \geq 1.$$

Here we introduce second order difference operator with polynomial factorial $\Delta_{\bar{\alpha}(k)} v(k) = v(k) - \alpha_1 k^{(p)} v(k-1) - \alpha_2 k^{(q)} v(k-2)$, where $\bar{\alpha}(k) = (\alpha_1 k^{(p)}, \alpha_2 k^{(q)})$ which generates $\bar{\alpha}(k)$ -Fibonacci sequence and its sum.

Definition 2.1. For $k \in [0, \infty)$, $\bar{\alpha}(k)$ -Fibonacci sequence is defined as

$$F_0 = 1, \quad F_1 = \alpha_1 k^{(p)}, \quad F_n = \alpha_1 [k - (n-1)]^{(p)} F_{n-1} + \alpha_2 [k - (n-2)]^{(q)} F_{n-2}, \quad n \geq 2 \quad (1)$$

Example 2.2. (i) Taking $k = 3, \alpha_1 = 2, \alpha_2 = 3, p = 2$ and $q = 3$ in (1), we get a Fibonacci sequence $\{1, 12, 66, \dots\}$.

(ii) When $k = 12, \alpha_1 = 2.22, \alpha_2 = 0.333, p = 3$ and $q = 2$ in (1), we have a Fibonacci sequence $\{1, 2930.40, 6440477.08, 10294565898.83, \dots\}$.

Similarly, one can obtain Fibonacci sequences corresponding to each pair $(\alpha_1 k^{(p)}, \alpha_2 k^{(q)}) \in \mathbb{R}^2$.

Definition 2.3. A second order difference operator with polynomial factorial on $v(k)$, denoted as $\Delta_{\bar{\alpha}(k)} v(k)$, where $\bar{\alpha}(k) = (\alpha_1 k^{(p)}, \alpha_2 k^{(q)})$, is defined as

$$\Delta_{\bar{\alpha}(k)} v(k) = v(k) - \alpha_1 k^{(p)} v(k-1) - \alpha_2 k^{(q)} v(k-2), \quad k \in [0, \infty), \quad (2)$$

and its inverse is defined as below;

$$\text{if } \Delta_{\bar{\alpha}(k)} v(k) = u(k), \text{ then we write } v(k) = \Delta_{\bar{\alpha}(k)}^{-1} u(k). \quad (3)$$

Lemma 2.4. *Let $v(k)$ be a functions of $k \in (-\infty, \infty)$. Then we obtained*

$$\Delta_{\bar{\alpha}(k)}^{-1} a^{sk} \left[1 - \frac{\alpha_1 k^{(p)}}{a} - \frac{\alpha_2 k^{(q)}}{a^{2s}} \right] = a^{sk}. \quad (4)$$

Proof. Taking $u(k) = a^{sk} \left[1 - \frac{\alpha_1 k^{(p)}}{a^s} - \frac{\alpha_2 k^{(q)}}{a^{2s}} \right]$ in (2), we obtained

$$\Delta_{\bar{\alpha}(k)} a^{sk} = a^{sk} \left[1 - \frac{\alpha_1 k^{(p)}}{a^s} - \frac{\alpha_2 k^{(q)}}{a^{2s}} \right]. \text{ Now (4) follows from (3).} \quad \square$$

Corollary 2.5. *If $\alpha_1 = 1 = \alpha_2$ in lemma 2.4, then we obtain*

$$\Delta_{\mu}^{-1} a^{sk} \left[1 - \frac{k^{(p)}}{a^s} - \frac{k^{(q)}}{a^{2s}} \right] = a^{sk}. \quad (5)$$

Proof. Taking $u(k) = a^{sk} \left[1 - \frac{k^{(p)}}{a^s} - \frac{k^{(q)}}{a^{2s}} \right]$ in (2), we have

$$\Delta_{\mu} a^{sk} = a^{sk} \left[1 - \frac{k^{(p)}}{a^s} - \frac{k^{(q)}}{a^{2s}} \right]. \text{ Now (5) follows from (3).} \quad \square$$

Lemma 2.6. *Let e^{-sk} be a function of $k \in (-\infty, \infty)$. Then we have*

$$\Delta_{\bar{\alpha}(k)}^{-1} e^{-sk} \left[1 - \alpha_1 k^{(p)} e^s - \alpha_2 k^{(q)} e^{2s} \right] = e^{-sk}. \quad (6)$$

Proof. The proof follows by assuming $a = e^{-1}$ in (4). \square

Corollary 2.7. *Let e^{-sk} be a function of $k \in (-\infty, \infty)$, then we obtained*

$$\Delta_{\mu}^{-1} e^{-sk} \left[1 - k^{(p)} e^s - k^{(q)} e^{2s} \right] = e^{-sk}. \quad (7)$$

Proof. The proof follows by taking $\alpha_1 = 1 = \alpha_2$ in lemma 2.6. \square

3. $\bar{\alpha}(k)$ -FIBONACCI SEQUENCE AND ITS SUM

In this section, we derive sum of the value of $\bar{\alpha}(k)$ -Fibonacci Sequence by inverse of $\bar{\alpha}(k)$ difference operator.

Theorem 3.1. *If $v(k) = \Delta_{\bar{\alpha}(k)}^{-1} u(k)$, $F_0 = 1$, $F_1 = \alpha_1 k^{(p)}$ and*

$F_{n+1} = \alpha_1 (k - n)^{(r)} F_n + \alpha_2 (k - (n - 1))^{(s)} F_{n-1}$, for $i = 0, 1, 2, \dots$ then we have

$$v(k) - F_{n+1} v(k - (n + 1)) - \alpha_2 (k - n)^{(q)} F_n v(k - (n + 2)) = \sum_{i=0}^n F_i u(k - i). \quad (8)$$

Proof. From (2) and (3), we arrive

$$v(k) = u(k) + \alpha_1 k^{(p)} v(k-1) + \alpha_2 k^{(q)} v(k-2). \quad (9)$$

Replacing k by $k-1$ and then substituting the value of $v(k-1)$ in (9), we get $v(k) = u(k) + \alpha_1 k^{(p)} [u(k-1) + \alpha_1 (k-1)^{(r)} v(k-2) + \alpha_2 (k-1)^{(s)} v(k-3)] + \alpha_2 k^{(p)} v(k-3)$ which gives

$$v(k) = F_0 u(k) + F_1 u(k-1) + F_2 v(k-2) + \alpha_2 (k-1)^{(s)} F_1 v(k-3), \quad (10)$$

where F_0 , F_1 and F_2 are given in (1).

Replacing k by $k-2$ in (9) and then substituting $v(k-2)$ in (10), we obtain

$$v(k) = F_0 u(k) + F_1 u(k-1) + F_2 u(k-2) + F_3 v(k-3) + \alpha_2 (k-2)^{(s)} F_2 v(k-4),$$

where F_3 is given in (1).

Repeating this process again and again, we get (8). \square

Corollary 3.2. If $\Delta_{\mu}^{-1} u(k) = v(k)$, $\mu = (k^{(p)}, k^{(q)})$, $F_0 = 1$, $F_1 = k^{(p)}$ and $F_{n+1} = (k-n)^{(r)} F_n + (k-(n-1))^{(s)} F_{n-1}$, for $i = 0, 1, 2, \dots$ then

$$v(k) - F_{n+1} v(k - (n+1)) - (k-n)^{(q)} F_n v(k - (n+2)) = \sum_{i=0}^n F_i u(k-i). \quad (11)$$

Proof. The proof follows by taking $\alpha_1 = 1 = \alpha_2$ in Theorem 3.1. \square

Corollary 3.3. If $v(k)$ is a closed form solution of the second order difference equation with polynomial factorial $\Delta_{\bar{\alpha}(k)} v(k) = a^{sk} [1 - \frac{\alpha_1 k^{(p)}}{a^s} - \frac{\alpha_2 k^{(q)}}{a^{2s}}]$, then we obtain $a^{sk} - F_{n+1} a^{s(k-(n+1))} - \alpha_2 (k-n)^{(q)} F_n a^{s(k-(n+2))}$

$$= \sum_{i=0}^n F_i a^{s(k-i)} [1 - \frac{\alpha_1 (k-i)^{(p)}}{a^s} - \frac{\alpha_2 (k-i)^{(q)}}{a^{2s}}]. \quad (12)$$

Proof. The proof follows by choosing $v(k) = a^{sk}$ in (8) and using (4). \square

The following example is an verification of (12).

Example 3.4. Taking $k = 3$, $n = 2$, $a = 2$, $\alpha_1 = 2$, $\alpha_2 = 3$, $s = 3$, $p = 2$ and $q = 3$ in (12), we get $2^9 - F_3 2^0 - 3 F_2 2^{-1} 1^3 = \sum_{i=0}^2 F_i a^{(3-i)} [1 - \frac{2(3-i)^{(2)}}{2^3} - \frac{3(3-i)^{(3)}}{2^6}] = 512$, where $F_0 = 1$, $F_1 = 12$, $F_2 = 66$, $F_3 = 0$.

MATLAB coding for the verification of Example 3.4

```
syms k; k = 3; syms n; n = 2; syms p; p = 2; syms q; q = 3;
syms s; s = 3; syms a1; a1 = 2; syms a2; a2 = 3; syms a; a = 2;
format bank
f1 = k * (k - 1)
f2 = f1 * (k - 1) * (k - 2) + k * (k - 1) * (k - 2)
f3 = f2 * (k - 2) * (k - 3) + f1 * (k - 1) * (k - 2) * (k - 3)
s1 = a(s * k)
s2 = f3 * a(s * (k - 3))
s3 = f2 * (k - 2) * (k - 3) * (k - 4) * a(s * (k - 4))
LHS = s1 - s2 - s3
u0 = a(s * k) - k * (k - 1) * a(s * (k - 1)) - k * (k - 1) * (k - 2) * a(s * (k - 2))
u1 = a(s * (k - 1)) - (k - 1) * (k - 2) * a(s * (k - 2)) - (k - 1) * (k - 2) * (k - 3) * a(s * (k - 3))
u2 = a(s * (k - 2)) - (k - 2) * (k - 3) * a(s * (k - 3)) - (k - 2) * (k - 3) * (k - 4) * a(s * (k - 4))
RHS = u0 + f1 * u1 + f2 * u2
```

Corollary 3.5. *Let e^{-sk} be a function of $k \in (-\infty, \infty)$. Then we have*

$$e^{-sk} - F_{n+1}e^{-s(k-(n+1))} - \alpha_2(k-n)^{(q)}F_n e^{-s(k-(n+2))} \\ = \sum_{i=0}^n F_i e^{-s(k-i)} [1 - \alpha_1(k-i)^{(p)}e^s - \alpha_2(k-i)^{(q)}e^{2s}]. \quad (13)$$

Proof. Taking $v(k) = e^{-sk}$ and applying (6) in (8), we get (13). \square

Example 3.6. *Taking $k = 2.88$, $n = 3$, $s = 3$, $\alpha_1 = 0.7$, $\alpha_2 = 0.5$, $p = 2$ and $q = 2$ in (??), then we obtain*

$$e^{-8.64} - F_4 e^{3.36} - (0.5)(-0.12)^{(2)}F_3 e^{6.36} = \sum_{i=0}^3 F_i e^{-3(2.88-i)} [1 - (0.7)(2.88-i)^{(2)}e^3 - (0.5)(2.88-i)^{(2)}e^6] = -97.43,$$

where $F_0 = 1$, $F_1 = 3.79$, $F_2 = 7.10$, $F_3 = 2.61$ and $F_4 = -0.13$.

Theorem 3.7. *Let $t \in \mathbb{N}(0)$. Then a closed form solution of the second order difference equation with polynomial factorial*

$$v(k) - \alpha_1 k^{(p)}v(k-1) - \alpha_2 k^{(q)}v(k-2) = [k^t - \alpha_1 k^{(p)}(k-1)^t - \alpha_2 k^{(q)}(k-2)^t] \text{ is} \\ \Delta_{\alpha(k)}^{-1} [k^t - \alpha_1 k^{(p)}(k-1)^t - \alpha_2 k^{(q)}(k-2)^t] = k^t \quad (14)$$

Proof. Taking $v(k) = k^t$ in (2) and using (3), we get (14). \square

Corollary 3.8. *Taking $t = 2$ in Theorem 3.7, we have*

$$\Delta_{\bar{\alpha}(k)}^{-1} [k^2 - \alpha_1 k^{(p)}(k-1)^2 - \alpha_2 k^{(q)}(k-2)^2] = k^2 \quad (15)$$

which is a closed form solution of the difference equation

$$\Delta_{\bar{\alpha}(k)} v(k) = k^2 - \alpha_1 k^{(p)}(k-1)^2 - \alpha_2 k^{(q)}(k-2)^2.$$

Proof. From (14), replacing $t = 2$, we get 15 □

Corollary 3.9. *If $v(k) = \Delta_{\bar{\alpha}(k)}^{-1} [k^t - \alpha_1 k^{(p)}(k-1)^t - \alpha_2 k^{(q)}(k-2)^t]$ is the closed form solution given in (14), then $v(k) - F_{n+1}v(k - (n+1) - \alpha_2(k-n)^{(q)}F_nv(k - (n+2)))$*

$$= \sum_{i=0}^n F_i [(k-i)^t - \alpha_1(k-i)^{(p)}[k-(i+1)]^t - \alpha_2(k-i)^{(q)}[k-(i+2)]^t]. \quad (16)$$

Proof. Taking $u(k) = k^t - \alpha_1 k^{(p)}(k-1)^t - \alpha_2 k^{(q)}(k-2)^t$ in Theorem 3.1, we have 16. □

Example 3.10. *Let $k = 11, n = 3, t = 3, p = 3, q = 4, \alpha_1 = 3, \alpha_2 = 2$ in Corollary (3.9). Then $\sum_{i=0}^3 F_i u(11-i) = v(11) - F_4 v(7) - \alpha_2 8^{(2)} F_3 v(6) = -10464437466361548$, where $u(k) = k^t - \alpha_1 k^{(p)}(k-1)^t - \alpha_2 k^{(q)}(k-2)^t$, $F_0 = 1, F_1 = 2970, F_2 = 6431040, F_3 = 9753670080$ and $F_4 = 9753670080$.*

Theorem 3.11. *If $v(k)$ is a closed form solution of second order difference equation*

$$\begin{aligned} &v(k) - \alpha_1 k^{(p)}v(k-1) - \alpha_2 k^{(q)}v(k-2) \\ &= k^t a^{sk} - \alpha_1 k^{(p)}(k-1)^t a^{s(k-1)} - \alpha_2 k^{(q)}(k-2)^t a^{s(k-2)}, \end{aligned}$$

then we have

$$\begin{aligned} v(k) - F_{n+1}v(k - [n+1]) - \alpha_2(k-n)^{(q)}v(k - [n+2]) &= \sum_{i=0}^n F_i [(k-i)^t a^{s(k-i)} \\ &- \alpha_1(k-i)^{(p)}[k-(i+1)]^t a^{s(k-(i+1))} - \alpha_2(k-i)^{(q)}[k-(i+2)]^t a^{s(k-(i+2))}]. \end{aligned} \quad (17)$$

Proof. Taking $u(k) = [k^t a^{sk} - \alpha_1 k^{(p)}(k-1)^t a^{s(k-1)} - \alpha_2 k^{(q)}(k-2)^t a^{s(k-2)}]$ in Theorem 3.1 and using (4), we get 17. □

Corollary 3.12. *A closed form solution of the second order difference equation with polynomial factorial $\Delta_{\bar{\alpha}(k)} v(k) = k^3 a^{sk} - \alpha_1 k^{(p)}(k-1)^3 a^{s(k-1)} - \alpha_2 k^{(p)}(k-2)^3 a^{s(k-2)}$ is $k^3 a^{sk}$ and hence we have*

$$v(k) - F_{n+1}v(k - (n + 1)) - \alpha_2(k - n)^{(q)}v(k - (n + 2)) = \sum_{i=0}^n F_i[(k - i)^3 a^{s(k-i)} \times \\ - \alpha_1(k - i)^{(p)}[k - (i + 1)]^3 a^{s(k-(i+1))} - \alpha_2(k - i)^{(q)}[k - (i + 2)]^3 a^{s(k-(i+2))}]. \quad (18)$$

Proof. The proof follows by taking $t = 3$ in Theorem 3.11. \square

Example 3.13. *Let $k = 13$, $a = 2$, $n = 3$, $s = 0.35$, $\alpha_1 = 0.8$, $\alpha_2 = 0.4$, $p = 3$, $q = 2$ in Corollary (3.12). Then we obtain*

$$v(13) - F_4v(9) - F_3(0.4)(10)^{(2)}v(8) = \sum_{i=0}^3 F_i[(13 - i)^3 2^{0.35(13-i)} - (0.8)(13 - i)^{(3)} \times \\ [13 - (i + 1)]^3 2^{0.35(13-(i+1))} - (0.4)(13 - i)^{(2)}[13 - (i + 2)]^3 2^{0.35(13-(i+2))}] = \\ -4427746334499175, \text{ where } F_0 = 1, F_1 = 1372.80, F_2 = 1449739.20, \\ F_3 = 1148265930.24 \text{ and } F_4 = 661464964343.04.$$

Corollary 3.14. *A closed form solution of the second order difference equation $v(k) - \alpha_1 k^{(p)}v(k - 1) - \alpha_2 k^{(q)}v(k - 2) = k^t e^{-sk} - \alpha_1 k^{(p)} \frac{(k - 1)^t}{e^{s(k-1)}} - \alpha_2 k^{(q)} \frac{(k - 2)^t}{e^{s(k-2)}}$ is given by*

$$v(k) - F_{n+1}v(k - (n + 1)) - \alpha_2 F_n(k - n)^{(q)}v(k - (n + 2)) = \sum_{i=0}^n F_i e^{-sk} \times \\ [(k - i)^t e^{is} - \alpha_1(k - i)^{(p)}[k - (i + 1)]^t e^{(i+1)s} - \alpha_2(k - i)^{(q)}[k - (i + 2)]^t e^{(i+2)s}]. \quad (19)$$

Proof. Taking $a = e^{-1}$ in (17), we get (19). \square

Corollary 3.15. *If $v(k) = \Delta_{\mu}^{-1}[ke^{-sk} - \alpha_1 k^{(p)}(k - 1)e^{-s(k-1)} - \alpha_2 k^{(q)}(k - 2)e^{-s(k-2)}]$ is the closed form solution given in (19), then*

$$v(k) - F_{n+1}v(k - (n + 1)) - \alpha_2 F_n(k - n)^{(q)}v(k - (n + 2)) = \sum_{i=0}^n F_i e^{-sk} \times \\ [(k - i)e^i - \alpha_1(k - i)^{(p)}[k - (i + 1)]e^{s(i+1)} - \alpha_2(k - i)^{(q)}[k - (i + 2)]e^{s(i+2)}]. \quad (20)$$

Proof. The proof follows by taking $t = 1$ in Corollary 3.14. \square

Theorem 3.16. Let $v(k)$ be a solution of the second order difference equation with

$$\begin{aligned} & \text{polynomial factorial } v(k) - \alpha_1 k^{(p)} v(k-1) - \alpha_2 k^{(q)} v(k-2) \\ & = k^{(t)} a^{sk} - \alpha_1 k^{(p)} (k-1)^{(t)} a^{s(k-1)} - \alpha_2 k^{(q)} (k-2)^{(t)} a^{s(k-2)}, \text{ then we have} \\ & v(k) - F_{n+1} v(k - [n+1]) - \alpha_2 (k-n)^{(q)} v(k - [n+2]) = \sum_{i=0}^n F_i a^{s(k-i)} \times \\ & \left[(k-i)^{(t)} - \alpha_1 (k-i)^{(p)} \frac{[k-(i+1)]^{(t)}}{a^s} - \alpha_2 (k-i)^{(q)} \frac{[k-(i+2)]^{(t)}}{a^{2s}} \right]. \end{aligned} \quad (21)$$

Proof. Taking $v(k) = k^{(t)} a^{sk}$ in Theorem 3.1 and using (4), we get 21. \square

Corollary 3.17. If $v(k)$ is the closed form solution given of (21), then

$$\begin{aligned} & k^{(2)} a^{sk} - F_{n+1} (k - (n+1))^{(2)} a^{s(k-(n+1))} - \alpha_2 (k-n)^{(q)} (k - (n+2))^{(2)} a^{s(k-(n+2))} = \\ & \sum_{i=0}^n F_i a^{s(k-i)} \left[(k-i)^{(2)} - \alpha_1 (k-i)^{(p)} \frac{(k-(i+1))^{(2)}}{a^s} - \alpha_2 (k-i)^{(q)} \frac{(k-(i+2))^{(2)}}{a^{2s}} \right] \end{aligned} \quad (22)$$

Proof. The proof follows by taking $t = 2$ in Theorem 3.16. \square

Example 3.18. Let $k = 7$, $a = 4$, $n = 2$, $s = 0.55$, $\alpha_1 = 0.5$, $\alpha_2 = 0.08$, $p = 3$, $q = 2$ in Corollary (3.17). Then we obtain

$$\begin{aligned} & v(7) - F_3 v(4) - (0.5) 5^{(2)} F_2 v(3) = \sum_{i=0}^2 F_i [(7-i)^{(2)} 4^{0.55(7-i)} - (0.5)(7-i)^{(3)} \times \\ & [7-(i+1)]^2 4^{0.55(7-(i+1))} - (0.08)(7-i)^{(2)} [7-(i+2)]^2 4^{0.55(7-(i+2))}] = -48558944.63, \\ & \text{where } F_0 = 1, F_1 = 105, F_2 = 6303.36, F_3 = 189352.80. \end{aligned}$$

Corollary 3.19. Let $v(k)$ be a solution of second order difference equation

$$v(k) - \alpha_1 k^{(p)} v(k-1) - \alpha_2 k^{(q)} v(k-2) = e^{-sk} [k^{(t)} - \alpha_1 k^{(p)} (k-1)^{(t)} e^s - \alpha_2 k^{(q)} (k-2)^{(t)} e^2].$$

Then we have

$$\begin{aligned} & v(k) - F_{n+1} v(k - [n+1]) - \alpha_2 (k-n)^{(q)} v(k - [n+2]) = \sum_{i=0}^n F_i e^{-s(k-i)} \times \\ & [(k-i)^{(t)} - \alpha_1 (k-i)^{(p)} [k-(i+1)]^{(t)} e^s - \alpha_2 (k-i)^{(q)} [k-(i+2)]^{(t)} e^{2s}]. \end{aligned} \quad (23)$$

Proof. Taking $a = e^{-1}$ in (3.16), we get (23). \square

Corollary 3.20. *A closed form solution of second order difference equation with polynomial factorial $\Delta_{\bar{\alpha}(k)} v(k) = e^{-sk} [k^{(2)} - \alpha_1 k^{(p)} (k-1)^{(2)} e^s - \alpha_2 k^{(q)} (k-2)^{(2)} e^{2s}]$ is $k^{(2)} e^{-sk}$ and hence we have*

$$v(k) - F_{n+1}v(k - (n+1)) - \alpha_2(k-n)^{(q)}v(k - (n+2)) = \sum_{i=0}^n F_i e^{-s(k-i)}$$

$$[(k-i)^{(2)} - \alpha_1(k-i)^{(p)}[k-(i+1)]^{(2)} e^s - \alpha_2(k-i)^{(q)}[k-(i+2)]^{(2)} e^{2s}]. \quad (24)$$

Proof. The proof follows by taking $t = 2$ in Corollary (3.19), we get (24). \square

Example 3.21. *Let $k = 12$, $n = 2$, $s = 0.64$, $a = 0.25$, $\alpha_1 = 2.22$, $\alpha_2 = 0.333$, $p = 3$, $q = 2$ in Corollary (3.20). Then we obtain*

$$v(12) - F_3v(9) - (0.333)F_2v(8) = \sum_{i=0}^2 F_i [(12-i)^{(2)}(0.25)^{0.64(12-i)} - (2.22)(12-i)^{(3)} \\ \times [12-(i+1)]^{(2)}(0.25)^{0.64(12-(i+1))} - (0.333)(12-i)^{(2)}[12-(i+2)]^{(2)}(0.25)^{0.64(12-(i+2))}] \\ = -2400227388.32, \text{ where } F_0 = 1, F_1 = 2930.40, F_2 = 6440477.08 \text{ and } F_3 = 10294565898.83.$$

Conclusion: We obtained summation formula to $\bar{\alpha}(k)$ -Fibonacci sequence by introducing $\bar{\alpha}(k)$ -difference operator and have derived certain results on closed and summation form solution of second order difference equation which will be used to our further research.

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