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Stability of Cubic-Quartic Functional Equation in Multi-Fuzzy Banach Spaces

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Abstract. In this paper we estabilish the Stability of Cubic-Quartic Functional Equations in Multi-Fuzzy Banach Spaces.

$$g(2a+b)+g(2a-b)=3g(a+b)+g(-a-b)+3g(a-b)+g(b-a)+18g(a) \\ +6g(-a)-3g(b)-3g(-b) \qquad (1)$$

Key words: Hyers-Ulam stability, Multi-Fuzzy-Banach Spaces, Cubic-Quartic Functional Equation, Fixed Point Method.

AMS Subject classification: Primary 39B82, Secondary 39B52, 46B99.

1. Introduction and Preliminaries

The first stability problem of functional equation was raised by S.M. Ulam [18] about seventy seven years ago. Since then, this question has attracted the attention of many researchers. The affirmtive solution to this question was given by D.H. Hyers [8] in 1941. In the year 1950, T. Aoki [2] generalized the Hyers theorem for additive mappings. The result of Hyers was generalized independently by Th.M.Rassias [16] for linear mappings by considering an unbounded Cauchy difference. In 1994, a further generalization of Th.M. Rassias theorem was obtained by P.Gavruta [7]. After that, the stability problem of several functional equations have been extensively investigated by a number of authors [?, 3, 17] on various spaces like, normed spaces, Banach spaces, Fuzzy normed spaces, Non-Archimedean space and etc.

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Sun-Young Jung, Choonkil Park and Dong Yun Shin [11] proved the Fuzzy stability of the Cubic-Quartic functional equation by using fixed point method. The multi-Banach space was first investigated by Dales and Polyakov [4]. Theory of multi-Banach spaces is similar to operator sequence space and has some connections with operator spaces and Banach spaces. In 2007 H.G. Dales and M.S. Moslehian [5] first proved the stability of mappings on multi-normed spaces and also gave some examples on multi-normed spaces. The stability of functional equations on multi-normed spaces was proved by many mathematicians (see, [?, 6, 10, 12]).

Definition 1.1. *[9] Let (E, N) be a fuzzy normed space. A multi-fuzzy norm on $\left\{E^k, k \in \mathbb{N}\right\}$ is a sequence N_k such that N_k is a fuzzy norm on $E^k (k \in \mathbb{N})$, $N_1(x,t) = N(x,t)$ for each $x \in E$ and $t \in \mathbb{R}$ and the following axioms are satisfied for each $k \in \mathbb{N}$ with $k \geq 2$:*

(1). *for each $\sigma \in, x \in E^k$ and $t \in \mathbb{R}$,*

$$N_k\left(A_\sigma(x), t\right) = N_k(x,t);$$

(2). *for each $\alpha = (\alpha_1, ..., \alpha_k) \in \mathbb{R}^k, x \in E^k$ and $t \in \mathbb{R}$,*

$$N_k\left(M_\alpha(x), t\right) \geq N_k\left(\max_{i \in \mathbb{N}_k} |\alpha_i| \, x, t\right);$$

(3). *for each $x_1, ..., x_k \in E$ and $t \in \mathbb{R}$,*

$$N_{k+1}\left((x_1, ..., x_k, 0), t\right) = N_k\left((x_1, ..., x_k), t\right);$$

(4). *for each $x_1, ..., x_k \in E$ and $t \in \mathbb{R}$,*

$$N_{k+1}\left((x_1, ..., x_k, x_k), t\right) = N_k\left((x_1, ..., x_k), t\right)$$

In such a case $\left\{(E^k, N_k), k \in \mathbb{N}\right\}$ is called a muti-fuzzy normed space.

$$\begin{aligned} Dg(a,b) = {} & g(2a+b) + g(2a-b) - 3g(a+b) - g(-a-b) - 3g(a-b) \\ & - g(b-a) - 18g(a) - 6g(-a) + 3g(b) + 3g(-b) \end{aligned} \tag{2}$$

Throughout this paper, assume that $\mathcal{X}$ be a linear space and let $(\mathcal{Y}^n, \|.\|_n)$ be a multi-Banach space.

Theorem 1.2. *Let $\phi : \mathcal{X}^{2k} \to [0, \infty)$ be a function such that there exists an $\mathcal{L} < 1$ with*

$$\phi\left(a_1, b_1, ..., a_k, b_k\right) \leq \frac{\mathcal{L}}{8}\phi\left(2a_1, 2b_1, ..., 2a_k, 2b_k\right)$$

for all $a_i, b_i \in \mathcal{X}$ where $i = 1, .., k$. Let $g : \mathcal{X} \to \mathcal{Y}$ be an odd mapping satisfies

$$N_k\left(\left(Dg(a_1, b_1), ..., Dg(a_k, b_k)\right), t\right) \geq \frac{t}{t + \phi(a_1, b_1, ..., a_k, b_k)} \tag{3}$$

for all $a_i, b_i \in \mathcal{X}$ where $i = 1, .., k$ and all $t > 0$. Then

$$\mathcal{C}(a) = N - \lim_{n\to\infty} 8^n g\left(\frac{a}{2^n}\right)$$

exists for each $a \in \mathcal{X}$ and defines a cubic mapping $\mathcal{C} : \mathcal{X} \to \mathcal{Y}$ such that

$$N_k\left(\left(g(a_1) - \mathcal{C}(a_1), ..., g(a_k) - \mathcal{C}(a_k)\right), t\right) \geq \frac{(16 - 16\mathcal{L})\, t}{(16 - 16\mathcal{L})\, t + \mathcal{L}\phi\left(a_1, 0, ..., a_k, 0\right)} \tag{4}$$

for all $a_i \in \mathcal{X}$ where $i = 1, .., k$ and all $t > 0$.

Proof. Letting $b_i = 0$ where $i = 1, .., k$ in (3), we get

$$N_k\left(\left(2g(2a_1) - 16g(a_1), ..., 2g(2a_k) - 16g(a_k)\right), t\right) \geq \frac{t}{t + \phi\left(a_1, 0, .., a_k, 0\right)} \tag{5}$$

for all $a_i \in \mathcal{X}$ where $i = 1, .., k$ and all $t > 0$. Consider the set $\mathcal{S} := \{h : \mathcal{X} \to \mathcal{Y}\}$ and introduce the generalized metric on $\mathcal{S}$:

$$d(h, l) = inf\left\{\mu \in \mathbb{R}_+ : N\left(h(a_1) - l(a_1), ..., h(a_k) - l(a_k), \mu t\right) \geq \frac{t}{t + \phi\left(a_1, 0, .., a_k, 0\right)}\right\}$$

where as usual $inf\phi = +\infty$. It is easy to prove that $(\mathcal{S}, d)$ is complete. See [13]. Now, we consider the linear mapping $\mathcal{J} : \mathcal{S} \to \mathcal{S}$ such that

$$\mathcal{J}h(a) = 8h\left(\frac{a}{2}\right)$$

for all $a \in \mathcal{X}$. Let $h, l \in \mathcal{S}$ be given such that $d(h, l) = \epsilon$. Then

$$N_k\left(\left(h(a_1) - l(a_1), ..., h(a_k) - l(a_k)\right), \epsilon t\right) \geq \frac{t}{t + \phi\left(a_1, 0, .., a_k, 0\right)}$$

for all $a_i \in \mathcal{X}$ where $i = 1, .., k$ and all $t > 0$. Hence

$$
\begin{aligned}
& N_k\left((\mathcal{J}h(a_1) - \mathcal{J}l(a_1), ..., \mathcal{J}h(a_k) - \mathcal{J}l(a_k)), \mathcal{L}\epsilon t\right) \\
&= N_k\left(\left(8h(\frac{a_1}{2}) - 8l(\frac{a_1}{2}), ..., 8h(\frac{a_k}{2}) - 8l(\frac{a_k}{2})\right), \mathcal{L}\epsilon t\right) \\
&= N_k\left(\left(h(\frac{a_1}{2}) - l(\frac{a_1}{2}), ..., h(\frac{a_k}{2}) - l(\frac{a_k}{2})\right), \frac{\mathcal{L}}{8}\epsilon t\right) \\
&\geq \frac{\frac{\mathcal{L}t}{8}}{\frac{\mathcal{L}t}{8} + \frac{\mathcal{L}}{8}\phi(a_1, 0, .., a_k, 0)} \\
&= \frac{t}{t + \phi(a_1, 0, .., a_k, 0)}
\end{aligned}
$$

for all $a_i \in \mathcal{X}$, where $i = 1, .., k$ and all $t > 0$. So $d(h, l) = \epsilon$ implies that $d(\mathcal{J}l, \mathcal{J}l) \leq \mathcal{L}\epsilon$. This means that

$$d(\mathcal{J}h, \mathcal{J}l) \leq \mathcal{L}d(h, l)$$

for all $h, l \in \mathcal{S}$. It follows from (5) that

$$N_k\left(\left(g(a_1) - 8(\frac{a_1}{2}), ..., g(a_k) - 8(\frac{a_k}{2})\right), \frac{\mathcal{L}}{16}t\right) \geq \frac{t}{t + \phi(a_1, 0, .., a_k, 0)}$$

for all $a_i \in \mathcal{X}$, where $i = 1, .., k$ and all $t > 0$. So $d(g, \mathcal{J}g) \leq \dfrac{\mathcal{L}}{16}$.

By Theorem 2.2 [14], there exists a cubic mapping $\mathcal{C} : \mathcal{X} \to \mathcal{Y}$ satisfying the following:

(1). $\mathcal{C}$ is a fixed point of $\mathcal{J}$, i.e

$$\mathcal{C}\left(\frac{a}{2}\right) = \frac{1}{8}\mathcal{C}(a) \tag{6}$$

for all $a \in \mathcal{X}$. The mapping $\mathcal{C}$ is a unique fixed point of $\mathcal{J}$ in the set $\mathcal{M} = \{h \in \mathcal{S} : d(g, h) < \infty\}$. This implies that C is a unique mapping satisfying (6) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N_k\left((g(a_1) - \mathcal{C}(a_1), ..., g(a_k) - \mathcal{C}(a_k)), \mu t\right) \geq \frac{t}{t + \phi(a_1, 0, .., a_k, 0)}$$

for all $a_i \in \mathcal{X}$, where $i = 1, .., k$ and all $t > 0$.

(2). $d(\mathcal{J}^n g, \mathcal{C}) \to 0$ as $n \to \infty$. This implies the equality

$$N - \lim_{n\to\infty} 8^n g\left(\frac{a}{2^n}\right) = \mathcal{C}(a) \qquad \text{for all} \quad a \in \mathcal{X}.$$

(3). $d(g,\mathcal{C}) \leq \dfrac{1}{1-\mathcal{L}} d(g,\mathcal{J}g)$, which implies the inequality

$$d(g,\mathcal{C}) \leq \frac{\mathcal{L}}{16-16\mathcal{L}}.$$

This implies that the inequality (4) holds. By (3),

$$N_k\left(\left(8^n Dg(\frac{a_1}{2},\frac{b_1}{2}),...,8^n Dg(\frac{a_k}{2},\frac{b_k}{2})\right), 8^n t\right) \geq \frac{t}{t+\phi\left(\frac{a_1}{2^n},\frac{b_1}{2^n},...,\frac{a_k}{2^n},\frac{b_k}{2^n}\right)}$$

for all $a_i, b_i \in \mathcal{X}$, where $i = 1,..,k$ all $t > 0$ and all $n \in \mathbb{N}$. So

$$N_k\left(\left(8^n Dg(\frac{a_1}{2},\frac{b_1}{2}),...,8^n Dg(\frac{a_k}{2},\frac{b_k}{2})\right), t\right) \geq \frac{\frac{t}{8^n}}{\frac{t}{8^n}+\frac{\mathcal{L}^n}{8^n}\phi\left(a_1,b_1,...,a_k,b_k\right)}$$

for all $a_i, b_i \in \mathcal{X}$, where $i = 1,..,k$ all $t > 0$ and all $n \in \mathbb{N}$. Since

$$\lim_{n\to\infty} \frac{\frac{t}{8^n}}{\frac{t}{8^n}+\frac{\mathcal{L}^n}{8^n}\phi\left(a_1,b_1,...,a_k,b_k\right)}$$

for all $a_i, b_i \in \mathcal{X}$, where $i = 1,..,k$ all $t > 0$.

$$N_k\left(\left(D\mathcal{C}(a_1,b_1),...,D\mathcal{C}(a_k,b_k)\right), t\right) = 1$$

for all $a_i, b_i \in \mathcal{X}$, where $i = 1,..,k$ all $t > 0$. Thus the mapping $\mathcal{C} : \mathcal{X} \to \mathcal{Y}$ is cubic, as desired. $\square$

Corollary 1.3. *Let $\vartheta \geq 0$ and let γ be a real number and $\gamma > 3$. Let $g : \mathcal{X} \to \mathcal{Y}$ be an odd mapping satisfying*

$$N_k\left(\left(Dg(a_1,b_1),...,Dg(a_k,b_k)\right), t\right) \geq \frac{t}{t+\vartheta\left(\|a_1\|^\gamma+\|b_1\|^\gamma,...,\|a_k\|^\gamma+\|b_k\|^\gamma\right)} \quad (7)$$

for all $a_i, b_i \in \mathcal{X}$, where $i = 1,..,k$ all $t > 0$. Then $\mathcal{C}(a) = N - \lim_{n\to\infty} 8^n g(\frac{a}{2^n})$ exists, and defines a cubic mapping $\mathcal{C} : \mathcal{X} \to \mathcal{Y}$ such that

$$N_k\left(\left(g(a_1)-\mathcal{C}(a_1),..,g(a_k)-\mathcal{C}(a_k)\right), t\right) \geq \frac{2\left(2^\gamma-8\right)t}{2(2^p-8)t+\vartheta\|a_1\|^\gamma,...,\|a_k\|^\gamma}$$

for all $a_i, b_i \in \mathcal{X}$, where $i = 1,..,k$ all $t > 0$.

Proof. The proof follows from Theorem 1.2 by taking

$$\phi\left(a_1,b_1,...,a_k,b_k\right) = \vartheta\left(\|a_1\|^\gamma+\|b_1\|^\gamma,...,\|a_k\|^\gamma+\|b_k\|^\gamma\right)$$

for all $a_i, b_i \in \mathcal{X}$, where $i = 1, .., k$ all $t > 0$. Then we can choose $L = 2^{3-\gamma}$ and we get the desired result. □

Theorem 1.4. *Let $\phi : \mathcal{X}^{2k} \to [0, \infty)$ be a function such that there exists an $\mathcal{L} < 1$ with*

$$\phi(a_1, b_1, ..., a_k, b_k) \leq \frac{\mathcal{L}}{16}\phi(2a_1, 2b_1, ..., 2a_k, 2b_k)$$

for all $a_i, b_i \in \mathcal{X}$ where $i = 1, .., k$. Let $g : \mathcal{X} \to \mathcal{Y}$ be an even mapping satisfying the inequality (3). Then

$$\mathcal{Q}(a) = N - \lim_{n\to\infty} 16^n g\left(\frac{a}{2^n}\right)$$

exists for each $a \in \mathcal{X}$ and defines a quartic function $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ such that

$$N_k\left((g(a_1) - \mathcal{Q}(a_1), ..., g(a_k) - \mathcal{Q}(a_k)), t\right) \geq \frac{(32 - 32\mathcal{L})\, t}{(32 - 32\mathcal{L})\, t + \mathcal{L}\phi(a_1, 0, ..., a_k, 0)} \tag{8}$$

for all $a_i \in \mathcal{X}$ where $i = 1, .., k$ and all $t > 0$.

Proof. Letting $b_i = 0$ where $i = 1, .., k$ in (3), we get

$$N_k\left((2g(2a_1) - 32g(a_1), ..., 2g(2a_k) - 32g(a_k)), t\right) \geq \frac{t}{t + \phi(a_1, 0, .., a_k, 0)} \tag{9}$$

for all $a_i \in \mathcal{X}$ where $i = 1, .., k$ and all $t > 0$. Consider the set $\mathcal{S} := \{h : \mathcal{X} \to \mathcal{Y}\}$ and introduce the generalized metric on $\mathcal{S}$:

$$d(h, l) = inf\left\{\mu \in \mathbb{R}_+ : N_k\left(h(a_1) - l(a_1), ..., h(a_k) - l(a_k), \mu t\right) \geq \frac{t}{t + \phi(a_1, 0, .., a_k, 0)}\right\}$$

where as usual $inf\phi = +\infty$. It is easy to show that $(\mathcal{S}, d)$ is complete. see [13]. Now, we consider the linear mapping $\mathcal{J} : \mathcal{S} \to \mathcal{S}$ such that

$$Jh(a) = 16h\left(\frac{a}{2}\right)$$

for all $a \in \mathcal{X}$. Let $h, l \in \mathcal{S}$ be given such that $d(h, l) = \epsilon$. Then

$$N_k\left((h(a_1) - l(a_1), ..., h(a_k) - l(a_k)), \epsilon t\right) \geq \frac{t}{t + \phi(a_1, 0, .., a_k, 0)}$$

for all $a_i \in \mathcal{X}$ where $i = 1, .., k$ and all $t > 0$. Hence

$$\begin{aligned}
&N_k\left(\left(\mathcal{J}h(a_1) - \mathcal{J}l(a_1), ..., \mathcal{J}h(a_k) - \mathcal{J}l(a_k)\right), \mathcal{L}\epsilon t\right)\\
&= N_k\left(\left(16h(\frac{a_1}{2}) - 16l(\frac{a_1}{2}), ..., 16h(\frac{a_k}{2}) - 16l(\frac{a_k}{2})\right), \mathcal{L}\epsilon t\right)\\
&= N_k\left(\left(h(\frac{a_1}{2}) - l(\frac{a_1}{2}), ..., h(\frac{a_k}{2}) - l(\frac{a_k}{2})\right), \frac{\mathcal{L}}{16}\epsilon t\right)\\
&\geq \frac{\frac{\mathcal{L}t}{16}}{\frac{\mathcal{L}t}{16} + \frac{\mathcal{L}}{16}\phi\left(a_1, 0, .., a_k, 0\right)}\\
&= \frac{t}{t + \phi\left(a_1, 0, .., a_k, 0\right)}
\end{aligned}$$

for all $a_i \in \mathcal{X}$, where $i = 1, .., k$ and all $t > 0$. So $d(h, l) = \epsilon$ implies that $d(\mathcal{J}h, \mathcal{J}l) \leq \mathcal{L}\epsilon$. This means that

$$d(\mathcal{J}h, \mathcal{J}l) \leq \mathcal{L}d(h, l)$$

for all $h, l \in \mathcal{S}$. It follows from (9) that

$$N_k\left(\left(g(a_1) - 16g(\frac{a_1}{2}), ..., g(a_k) - 16g(\frac{a_k}{2})\right), \frac{\mathcal{L}}{32}t\right) \geq \frac{t}{t + \phi\left(a_1, 0, .., a_k, 0\right)}$$

for all $a_i \in \mathcal{X}$, where $i = 1, .., k$ and all $t > 0$. So $d(g, \mathcal{J}g) \leq \frac{\mathcal{L}}{32}$.
By Theorem 2.2 [14], there exists a quartic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ satisfying the following:
(1). $\mathcal{Q}$ is a fixed point of $\mathcal{J}$, i.e

$$\mathcal{Q}\left(\frac{a}{2}\right) = \frac{1}{16}\mathcal{Q}(a) \tag{10}$$

for all $a \in \mathcal{X}$. The mapping $\mathcal{Q}$ is a unique fixed point of $\mathcal{J}$ in the set $\mathcal{M} = \{h \in \mathcal{S} : d(g, h) < \infty\}$. This implies that $\mathcal{Q}$ is a unique mapping satisfying (10) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N_k\left(\left(g(a_1) - \mathcal{Q}(a_1), ..., g(a_k) - \mathcal{Q}(a_k)\right), \mu t\right) \geq \frac{t}{t + \phi\left(a_1, 0, .., a_k, 0\right)}$$

for all $a_i \in \mathcal{X}$, where $i = 1, .., k$ and all $t > 0$.
(2). $d(\mathcal{J}^n g, \mathcal{Q}) \to 0$ as $n \to \infty$. This implies the equality

$$N - \lim_{n\to\infty} 16^n g\left(\frac{a}{2^n}\right) = \mathcal{Q}(a)$$

for all $a \in \mathcal{X}$.
(3). $d(g,\mathcal{Q}) \leq \frac{1}{1-L} d(g,\mathcal{J}g)$, which implies the inequality

$$d(g,\mathcal{Q}) \leq \frac{\mathcal{L}}{32-32\mathcal{L}}.$$

This implies that the inequality (8) holds. The rest of the proof is similar to the proof of Theorem1.2. □

Corollary 1.5. *Let $\vartheta \geq 0$ and let γ be a real number and $\gamma > 4$. Let $g : \mathcal{X} \to \mathcal{Y}$ be an even mapping satisfying*

$$N_k\left((Dg(a_1,b_1),...,Dg(a_k,b_k)),t\right) \geq \frac{t}{t+\vartheta\left(\|a_1\|^\gamma+\|b_1\|^\gamma,...,\|a_k\|^\gamma+\|b_k\|^\gamma\right)} \tag{11}$$

for all $a_i, b_i \in \mathcal{X}$, where $i = 1,..,k$ all $t > 0$. Then $\mathcal{Q}(a) = N - \lim_{n\to\infty} 16^n g(\frac{a}{2^n})$ exists, and defines a quartic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ such that

$$N_k\left((g(a_1)-\mathcal{Q}(a_1),..,g(a_k)-\mathcal{Q}(a_k)),t\right) \geq \frac{2\left(2^\gamma-16\right)t}{2(2^\gamma-16)t+\vartheta\left\|a_1\right\|^\gamma,...,\left\|a_k\right\|^\gamma}$$

for all $a_i, b_i \in \mathcal{X}$, where $i = 1,..,k$ all $t > 0$.

The proof follows from Theorem 1.4 by taking

$$\phi\left(a_1,b_1,...,a_k,b_k\right) = \vartheta\left(\|a_1\|^\gamma+\|b_1\|^\gamma,...,\|a_k\|^\gamma+\|b_k\|^\gamma\right)$$

for all $a_i, b_i \in a$, where $i = 1,..,k$ all $t > 0$. Then we can choose $\mathcal{L} = 2^{4-\gamma}$ and we get the desired result.

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