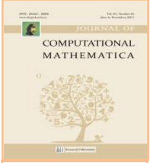
J. Comp.Matha. Vol.1(2), (2017). ISSN: 2456–8686

Original Research Article

Journal of Computational Mathematica

Journal homepage: www.shcpub.edu.in



ISSN: 2456-8686

On the Oscillation of Impulsive Vector Partial Differential Equations with Damping Term

[1]V.Sadhasivam and [2]T. Raja

Received on 28 August 2017, Accepted on 18 November 2017

Abstract. This article is an attempt to study the class of impulsive vector partial functional differential equations with continuous distributed deviating arguments. The main aim of this paper is to present some sufficient conditions for the H-oscillation of solutions, using impulsive differential inequalities and an averaging technique with two different boundary conditions. Our main results are point up with a suitable example.

Key words: Neutral partial differential equations, Oscillation, Vector, Impulse, Distributed deviating arguments.

AMS Subject classification: 34A37, 35L70, 35R10, 35R12.

1. Introduction

Impulsive differential equations arise in many biological, chemical, physical systems and electrical networks. They appear in these systems as a natural description of observed evolution phenomena, in several real world problems. The theory of impulsive differential equations is much richer than the corresponding theory of differential equations without impulse effects and has many real world applications, since many models involve forces acting abruptly, almost instantly, and at different times. The theory of impulsive differential equations has its origins in Mil'man and Mishkys paper, in [17].

[1]Corresponding Author: E-mail: ovsadha@gmail.com, trmaths19@gmail.com

[1,2]PG and Research Department of Mathematics, Thiruvalluvar Government Arts College, Rasipuram, Namakkal District, Tamil Nadu, India.

Early work on the oscillation theory of impulsive differential equations appeared in 1989 [6]. The first paper on impulsive partial differential equations [4] was published in 1991. Following these early results, several authors worked on the oscillatory behavior of impulsive partial differential equations with delays [5, 13, 14, 15, 16, 24, 25, 26, 30, 31]

In 1970, Dom*šlak* introduced the concept of H-oscillation to study the oscillation of the solutions of vector differential equations, where H is a unit vector in $\mathbb{R}^M$. We refer the reader to [2, 3, 8] for vector ordinary differential equations, [12, 18, 19, 20, 21, 23] for vector partial differential equations and [11, 22] for impulsive vector partial differential equations. For the essential background on the oscillation theory of differential equations, we refer the reader to the monographs [1, 10, 27, 28, 29, 32] and the references cited therein. There has been very little work on the study of impulsive vector partial differential equations with functional arguments. This scarcity has been the motivation that has led us to attempt to initiate a research effort and make some progress, in the study of impulsive nonlinear vector partial differential equations with continuous distributed deviating arguments.

In this paper, we consider impulsive nonlinear neutral delay vector partial differential equations with damping term of the form

$$\begin{cases} \dfrac{\partial}{\partial t}\Big[p(t)\dfrac{\partial}{\partial t}\left(U(x,t)+g(t)U(x,\rho(t))\right)\Big]+r(t)\dfrac{\partial}{\partial t}\left(U(x,t)+g(t)U(x,\rho(t))\right) \\ +\displaystyle\int_c^d q(x,t,\zeta)U(x,\tau(t,\zeta))d\eta(\zeta)=a(t)\Delta U(x,t) \\ +\displaystyle\int_c^d a(t,\zeta)\Delta U(x,\theta(t,\zeta))d\eta(\zeta)+F(x,t), \qquad \text{for } t\neq t_j \\ U(x,t_j^+)=a_j\left(x,t_j,U(x,t_j)\right), \\ \dfrac{\partial U(x,t_j^+)}{\partial t}=b_j\left(x,t_j,\dfrac{\partial U(x,t_j)}{\partial t}\right), \qquad \text{for} \quad t=t_j,\ j=1,2,\cdots, \\ \qquad\qquad\qquad\qquad\qquad \text{and} \quad (x,t)\in\Omega\times\mathbb{R}_+\equiv G, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a piecewise smooth boundary $\partial\Omega$ and Δ is the Laplacian in the Euclidean space $\mathbb{R}^N$. Moreover, we consider the following boundary conditions:

$$\frac{\partial}{\partial \gamma}U(x,t)+\mu(x,t)U(x,t)=0, \quad (x,t)\in \partial\Omega\times\mathbb{R}_+ \tag{2}$$

$$U(x,t)=0, \quad (x,t)\in \partial\Omega\times\mathbb{R}_+ \tag{3}$$

where γ is the outward unit normal vector to $\partial\Omega$, $\mu(x,t)\in C\left(\partial\Omega\times\mathbb{R}_+,\mathbb{R}_+\right)$ and $\mathbb{R}_+=[0,+\infty)$.

Next, we present the following set of conditions which we assume to hold, throughout the paper.

(H₁) $r(t)\in C(\mathbb{R}_+,\mathbb{R})$, $p(t)\in C^1\left(\mathbb{R}_+,(0,+\infty)\right)$ with $p'(t)\geqslant 0$, $\displaystyle\int_{t_0}^{+\infty}\frac{1}{P(s)}ds=+\infty$, where $P(t)=\exp\left(\displaystyle\int_{t_0}^{t}\frac{p'(s)+r(s)}{p(s)}ds\right)$, and $g(t)\in C^2(\mathbb{R}_+\times[c,d],\mathbb{R}_+)$.

(H₂) $q(x,t,\zeta)\in C\left(\bar{G}\times[c,d],\mathbb{R}_+\right)$, $Q(t,\zeta)=\displaystyle\min_{x\in\bar{\Omega}}q(x,t,\zeta)$, $\tau(t,\zeta)$, $\theta(t,\zeta)\in C\left(\mathbb{R}_+\times[c,d],\mathbb{R}\right)$, $\tau(t,\zeta)\le t$, $\theta(t,\zeta)\le t$ for $\zeta\in[c,d]$, $\tau(t,\zeta)$ and $\theta(t,\zeta)$ are non-decreasing with respect to t and ζ respectively and

$$\liminf_{t\to+\infty,\ \zeta\in[c,d]}\tau(t,\zeta)=\liminf_{t\to+\infty,\ \zeta\in[c,d]}\theta(t,\zeta)=+\infty.$$

There exists a function $\sigma(t)\in C^1(\mathbb{R}_+,\mathbb{R}_+)$ satisfying $\sigma(t)\leqslant\tau(t,c)$, with $\sigma'(t)>0$ and $\displaystyle\lim_{t\to+\infty}\sigma(t)=+\infty$.

(H₃) $a(t,\zeta)\in C(\mathbb{R}_+\times[c,d],\mathbb{R})$, $a(t)\in PC(\mathbb{R}_+,\mathbb{R}_+)$, where PC denotes the class of functions which are piecewise continuous in t with discontinuities of the first kind, only at $t=t_j,\ j=1,2,\cdots$, and left continuous at $t=t_j,\ j=1,2,\cdots$.

(H₄) $\rho(t)\in C(\mathbb{R}_+,\mathbb{R})$, $\displaystyle\lim_{t\to+\infty}\rho(t)=+\infty$, $\eta(\zeta):[c,d]\to\mathbb{R}$ is nondecreasing, integral of (1) is a Stieltjes integral, $F\in C(\bar{G},\mathbb{R}^M)$, $f_H(x,t)\in C(\bar{G},\mathbb{R})$ and $\int_\Omega f_H(x,t)dx\le 0$.

(H₅) All the components of $U(x,t)$ and its derivative $\frac{\partial}{\partial t}U(x,t)$ are piecewise continuous in t with discontinuities of the first kind, only at $t=t_j,\ j=1,2,\cdots$, and left continuous at $t=t_j$

$$U(x,t_j)=U(x,t_j^-),\quad \frac{\partial}{\partial t}U(x,t_j)=\frac{\partial}{\partial t}U(x,t_j^-),\quad j=1,2,\cdots.$$

(H₆) $a_j,\ b_j\in PC(\bar{G}\times\mathbb{R},\mathbb{R})$ for $j=1,2,\cdots$, and there exist constants $\alpha_j,\alpha_j^*,\beta_j,\beta_j^*$ such that for $j=1,2,\cdots$,

$$\alpha_j^* \leqslant \frac{a_j\left(x, t_j, U(x, t_j)\right)}{U(x, t_j)} \leqslant \alpha_j, \qquad \beta_j^* \leqslant \frac{b_j\left(x, t_j, \dfrac{\partial U(x,t_j)}{\partial t}\right)}{\dfrac{\partial U(x,t_j)}{\partial t}} \leqslant \beta_j.$$

In Section 2, we present the definitions and introduce the notation we will use through the paper. In Section 3, we discuss the H-oscillation of problem (1)-(2). In Section 4, we discuss the H-oscillation of problem (1)-(3). In Section 5 we present an example to point up the main result.

2. Preliminaries

In this section, we present some definitions and review some noteworthy results, from the literature which we will use throughout the paper.

Definition 2.1 (See [32]). *By a* **solution** *of* (1)-(2)*[*(3)*] we mean a function* $U(x,t) \in C^2(\overline{\Omega} \times [t_1, +\infty), \mathbb{R}^M) \cap C^1(\overline{\Omega} \times [\widehat{t_1}, +\infty), \mathbb{R}^M)$ *which satisfies* (1)*, where* $t_1 := \min\left\{0, \min_{1 \leqslant i \leqslant m}\left[\inf_{t \geqslant 0} \theta(t,\zeta)\right], \left\{\inf_{t \geq 0} \rho(t)\right\}\right\}$ *and* $\widehat{t_1} := \min\left\{0, \min_{\zeta \in [c,d]}\left\{\inf_{t \geqslant 0} \tau(t,\zeta)\right\}\right\}$.

Now based on this definition of a solution, we can precisely define what we mean by *H-oscillation*.

Definition 2.2 (See [32]). *Let H be a fixed unit vector in $\mathbb{R}^M$. A solution $U(x,t)$ of* (1) *is said to be H-oscillatory in G if the inner product $\langle U(x,t), H\rangle$ has a zero in $\Omega \times [t, +\infty)$ for $t > 0$. Otherwise it is a H-nonoscillatory.*

Next, we state two results which will help us establish our results.

Lemma 2.3 (See [6]). *If X and Y are nonnegative, then*

$$X^\delta - \delta X Y^{\delta - 1} + (\delta - 1) Y^\delta \geqslant 0, \quad \text{if } \delta > 1$$

$$X^\delta - \delta X Y^{\delta - 1} - (1 - \delta) Y^\delta \leqslant 0, \quad \text{if } 0 < \delta < 1.$$

In both cases, the equality holds, if and only if $X = Y$. It is known [28] *that the first eigenvalue λ_0 of the problem*

$$\begin{cases} \Delta \nu(x) + \lambda \nu(x) = 0 & \text{in } \Omega \\ \nu(x) = 0 & \text{on } \partial\Omega \end{cases}$$

is positive and the corresponding eigenfunction $\varphi(x)$ is positive in Ω.

For convenience, we introduce the following notations:

$$\begin{aligned} u_H(x,t) &= \langle U(x,t), H\rangle, \quad G(t) = g_0 \int_c^d Q(t,\zeta)d\eta(\zeta), \\ V_H(t) &= \frac{1}{|\Omega|}\int_\Omega u_H(x,t)dx, \quad \tilde{V}_H(t) = K_\varphi \int_\Omega u_H(x,t)\varphi(x)dx \end{aligned}$$

where $g_0 = 1 - g(\tau(t,\zeta))$, $|\Omega| = \int_\Omega dx$, $K_\varphi = \left(\int_\Omega \varphi(x)dx\right)^{-1}$.

3. H-Oscillations of (1)-(2)

In this section, we establish sufficient conditions for the H-oscillation of all solutions of (1)-(2).

Lemma 3.1. *Let H be a fixed unit vector in $\mathbb{R}^M$ and $U(x,t)$ be a solution of* (1).

(i) *If $u_H(x,t)$ is eventually positive, then $u_H(x,t)$ satisfies the scalar impulsive partial differential inequality*

$$\begin{cases} \dfrac{\partial}{\partial t}\left(p(t)\dfrac{\partial}{\partial t}\left(u_H(x,t)+g(t)u_H(x,\rho(t))\right)\right) \\ +r(t)\dfrac{\partial}{\partial t}\left(u_H(x,t)+g(t)u_H(x,\rho(t))\right) + \displaystyle\int_c^d Q(t,\zeta)u_H(x,\tau(t,\zeta))d\eta(\zeta) \\ -a(t)\Delta u_H(x,t) - \displaystyle\int_c^d a(t,\zeta)\Delta u_H(x,\theta(t,\zeta))d\eta(\zeta) \leqslant f_H(x,t), \qquad t\neq t_j \\ \alpha_j^* \leqslant \dfrac{u_H(x,t_j^+)}{u_H(x,t_j)} \leqslant \alpha_j, \quad \beta_j^* \leqslant \dfrac{u'_H(x,t_j^+)}{u'_H(x,t_j)} \leqslant \beta_j, \quad j=1,2,\cdots. \end{cases} \tag{4}$$

(ii) *If $u_H(x,t)$ is eventually negative, then $u_H(x,t)$ satisfies the scalar impulsive partial differential inequality*

$$\begin{cases} \dfrac{\partial}{\partial t}\left(p(t)\dfrac{\partial}{\partial t}\left(u_H(x,t)+g(t)u_H(x,\rho(t))\right)\right) \\ +r(t)\dfrac{\partial}{\partial t}\left(u_H(x,t)+g(t)u_H(x,\rho(t))\right) + \displaystyle\int_c^d Q(t,\zeta)u_H(x,\tau(t,\zeta))d\eta(\zeta) \\ -a(t)\Delta u_H(x,t) - \displaystyle\int_c^d a(t,\zeta)\Delta u_H(x,\theta(t,\zeta))d\eta(\zeta) \geqslant f_H(x,t), \qquad t\neq t_j \\ \alpha_j^* \geqslant \dfrac{u_H(x,t_j^+)}{u_H(x,t_j)} \geqslant \alpha_j, \quad \beta_j^* \geqslant \dfrac{u'_H(x,t_j^+)}{u'_H(x,t_j)} \geqslant \beta_j, \quad j=1,2,\cdots. \end{cases} \tag{5}$$

Proof. (i) Let $u_H(x,t)$ be eventually positive.

Case:(i) $t\neq t_j,\ j=1,2,\cdots$. Taking the inner product of (1) and H, we get

$$
\begin{aligned}
&\frac{\partial}{\partial t}\Big(p(t)\frac{\partial}{\partial t}\left(\langle U(x,t),H\rangle + g(t)\langle U(x,\rho(t)),H\rangle\right)\Big)\\
&+r(t)\frac{\partial}{\partial t}\left(\langle U(x,t),H\rangle + g(t)\langle U(x,\rho(t)),H\rangle\right) + \int_c^d q(x,t,\zeta)\langle U(x,\tau(t,\zeta)),H\rangle d\eta(\zeta)\\
&= a(t)\Delta\langle U(x,t),H\rangle + \int_c^d a(t,\zeta)\Delta\langle U(x,\theta(t,\zeta)),H\rangle d\eta(\zeta) + \langle F(x,t),H\rangle,
\end{aligned}
$$

that is

$$
\begin{cases}
\frac{\partial}{\partial t}\Big(p(t)\frac{\partial}{\partial t}\left(u_H(x,t) + g(t)u_H(x,\rho(t))\right)\Big)\\
+r(t)\frac{\partial}{\partial t}\left(u_H(x,t) + g(t)u_H(x,\rho(t))\right) + \int_c^d q(x,t,\zeta)u_H(x,\tau(t,\zeta))d\eta(\zeta)\\
= a(t)\Delta u_H(x,t) + \int_c^d a(t,\zeta)\Delta u_H(x,\theta(t,\zeta))d\eta(\zeta) + f_H(x,t).
\end{cases} \tag{6}
$$

Using condition (H_2), we have

$$
\int_c^d q(x,t,\zeta)u_H(x,\tau(t,\zeta))d\eta(\zeta) \geqslant \int_c^d Q(t,\zeta)u_H(x,\tau(t,\zeta))d\eta(\zeta). \tag{7}
$$

From (6) and (7), it follows that

$$
\begin{cases}
\frac{\partial}{\partial t}\Big(p(t)\frac{\partial}{\partial t}\left(u_H(x,t) + g(t)u_H(x,\rho(t))\right)\Big)\\
+r(t)\frac{\partial}{\partial t}\left(u_H(x,t) + g(t)u_H(x,\rho(t))\right) + \int_c^d Q(t,\zeta)u_H(x,\tau(t,\zeta))d\eta(\zeta)\\
-a(t)\Delta u_H(x,t) - \int_c^d a(t,\zeta)\Delta u_H(x,\theta(t,\zeta))d\eta(\zeta) \leqslant f_H(x,t),\ t\neq t_j.
\end{cases} \tag{8}
$$

Case:(ii) $t = t_j,\ j = 1,2,\cdots$. Taking the inner product of (1) and H, and using (H_6), we obtain

$$
\begin{aligned}
\alpha_j^* \leqslant \frac{U(x,t_j^+)}{U(x,t_j)} \leqslant \alpha_j, \qquad & \beta_j^* \leqslant \frac{U'(x,t_j^+)}{U'(x,t_j)} \leqslant \beta_j\\
\alpha_j^* \leqslant \frac{\langle U(x,t_j^+),H\rangle}{\langle U(x,t_j),H\rangle} \leqslant \alpha_j, \qquad & \beta_j^* \leqslant \frac{\langle U'(x,t_j^+),H\rangle}{\langle U'(x,t_j),H\rangle} \leqslant \beta_j
\end{aligned}
$$

that is

$$
\alpha_j^* \leqslant \frac{u_H(x,t_j^+)}{u_H(x,t_j)} \leqslant \alpha_j, \quad \beta_j^* \leqslant \frac{u'_H(x,t_j^+)}{u'_H(x,t_j)} \leqslant \beta_j, \quad j = 1,2,\cdots. \tag{9}
$$

Therefore, combining (8) and (9) we immediately obtain (4), which shows that $u_H(x,t)$ satisfies the scalar impulsive partial differential inequality (4).

(ii) The proof is similar to that of (i) and we omit it. The proof is complete. □

Let H be a fixed unit vector in $\mathbb{R}^M$. The inner product of boundary conditions (2) [(3)] and H yields the following boundary conditions:

$$\frac{\partial}{\partial \gamma} u_H(x,t) + \mu(x,t)u_H(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}_+, \tag{$2'$}$$

$$u_H(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}_+. \tag{$3'$}$$

Lemma 3.2. *Let H be a fixed unit vector in $\mathbb{R}^M$. If the scalar impulsive partial differential inequality* (4) *has no eventually positive solutions and the scalar impulsive partial differential inequality* (5) *has no eventually negative solutions, satisfying the boundary conditions* ($2'$) [($3'$)], *then every solution $U(x,t)$ of* (1)-(2)[(3)] *is H-oscillatory in $\Omega \times \mathbb{R}_+$.*

Proof. Suppose to the contrary that there is a H-nonoscillatory solution $U(x,t)$ of (1)-(2)[(3)] in G, then $u_H(x,t)$ is eventually positive or eventually negative. If $u_H(x,t)$ is eventually positive, by Lemma 3.1, easily, we obtain that $u_H(x,t)$ satisfies the scalar impulsive partial differential inequality (4). On the other hand, it is easy to see that $u_H(x,t)$ satisfies the boundary conditions ($2'$)[($3'$)]. This is a contradiction to the hypothesis.

Similarly, if $u_H(x,t)$ is eventually negative using Lemma 3.1, easily, we obtain that $u_H(x,t)$ satisfies the scalar impulsive partial differential inequality (5). It is obvious that $u_H(x,t)$ satisfies, the boundary conditions ($2'$)[($3'$)]. This is a contradiction. The proof is complete. □

Theorem 3.3. *Let H be a fixed unit vector in $\mathbb{R}^M$. If the impulsive differential inequality*

$$\begin{cases} (p(t)Z'_H(t))' + r(t)Z'_H(t) + G(t)Z_H(\sigma(t)) \leqslant 0, \quad t \neq t_j \\ \alpha_j^* \leqslant \dfrac{Z_H(t_j^+)}{Z_H(t_j)} \leqslant \alpha_j, \quad \beta_j^* \leqslant \dfrac{Z'_H(t_j^+)}{Z'_H(t_j)} \leqslant \beta_j, \quad j = 1, 2, \cdots, \end{cases} \tag{10}$$

has no eventually positive solutions and the impulsive differential inequality

$$\begin{cases} (p(t)Z'_H(t))' + r(t)Z'_H(t) + G(t)Z_H(\sigma(t)) \geqslant 0, \quad t \neq t_j \\ \alpha_j^* \geqslant \dfrac{Z_H(t_j^+)}{Z_H(t_j)} \geqslant \alpha_j, \quad \beta_j^* \geqslant \dfrac{Z'_H(t_j^+)}{Z'_H(t_j)} \geqslant \beta_j, \quad j = 1, 2, \cdots, \end{cases} \tag{11}$$

has no eventually negative solutions, then every solution $U(x,t)$ of (1)-(2) *is H-oscillatory in $\Omega \times \mathbb{R}_+$.*

Proof. Suppose to the contrary that there exists a solution $U(x,t)$ of (1)-(2) which is not H-oscillatory in $\Omega\times\mathbb{R}_+$. Without loss of generality, we may assume that $u_H(x,t)>0$ in $\Omega\times[t_0,+\infty)$, for some $t_0>0$. By the assumption that there exists a $t_1>t_0$ such that $\rho(t)\geq t_0$ for $t\geqslant t_0$, $\tau(t,\zeta)\geqslant t_0$ and $\theta(t,\zeta)\geqslant t_0$ for $(t,\zeta)\in[t_1,+\infty)\times[c,d]$, we have that

$$u_H(x,\rho(t))>0,\quad u_H(x,\tau(t,\zeta))>0\quad\text{ and }\quad u_H(x,\theta(t,\zeta))>0,$$

for $x\in\Omega$, $t\in[t_1,+\infty)$, $\zeta\in[c,d]$.

For $t\geqslant t_0$ and $t\neq t_j$ for $j=1,2,\cdots$, we multiply both sides of inequality (4) by $\dfrac{1}{|\Omega|}$ and integrate with respect to x over the domain Ω to attain

$$\begin{cases}\dfrac{d}{dt}\left(p(t)\dfrac{d}{dt}\left(\dfrac{1}{|\Omega|}\displaystyle\int_\Omega u_H(x,t)dx+\dfrac{1}{|\Omega|}\int_\Omega g(t)u_H(x,\rho(t))dx\right)\right)\\ +r(t)\dfrac{d}{dt}\left(\dfrac{1}{|\Omega|}\displaystyle\int_\Omega u_H(x,t)dx+\dfrac{1}{|\Omega|}\int_\Omega g(t)u_H(x,\rho(t))dx\right)\\ +\dfrac{1}{|\Omega|}\displaystyle\int_\Omega\int_c^d Q(t,\zeta)u_H(x,\tau(t,\zeta))d\eta(\zeta)dx-a(t)\dfrac{1}{|\Omega|}\int_\Omega\Delta u_H(x,t)dx\\ -\dfrac{1}{|\Omega|}\displaystyle\int_\Omega\int_c^d a(t,\zeta)\Delta u_H(x,\theta(t,\zeta))d\eta(\zeta)dx\leqslant\dfrac{1}{|\Omega|}\int_\Omega f_H(x,t)dx,\quad t\neq t_j\end{cases}\tag{12}$$

Using Green's formula and boundary condition $(2')$, we have that

$$\int_\Omega\Delta u_H(x,t)dx=\int_{\partial\Omega}\frac{\partial}{\partial\gamma}u_H(x,t)dS=-\int_{\partial\Omega}\mu(x,t)u_H(x,t)dS\leqslant 0,\tag{13}$$

and

$$\begin{aligned}\int_\Omega\Delta u_H(x,\theta(t,\zeta))dx&=\int_{\partial\Omega}\frac{\partial}{\partial\gamma}u_H(x,\theta(t,\zeta))dS\\ &=-\int_{\partial\Omega}\mu(x,\theta(t,\zeta))u_H(x,\theta(t,\zeta))dS\leqslant 0,\quad t\geqslant t_0\end{aligned}\tag{14}$$

where dS is the surface component on $\partial\Omega$. Furthermore, by (H_4), $\int_\Omega f_H(x,t)dx\leqslant 0$. Combining (12)-(14) we get

$$\begin{aligned}&\frac{d}{dt}\left[p(t)\frac{d}{dt}\left(V_H(t)+g(t)V_H(\rho(t))\right)\right]+r(t)\frac{d}{dt}\left(V_H(t)+g(t)V_H(\rho(t))\right)\\ &+\int_c^d Q(t,\zeta)V_H(\tau(t,\zeta))d\eta(\zeta)\leqslant 0,\quad t\geq t_0.\end{aligned}$$

Setting $Z_H(t)=V_H(t)+g(t)V_H(\rho(t))$, we have

$$(p(t)Z'_H(t))' + r(t)Z'_H(t) + \int_c^d Q(t,\zeta)V_H(\tau(t,\zeta))d\eta(\zeta) \leqslant 0. \tag{15}$$

Clearly $Z_H(t) > 0$ for $t \geqslant t_1$. Next we prove that $Z'_H(t) > 0$ for $t \geqslant t_2$. In fact assume there exists $T \geqslant t_2$ such that $Z'_H(T) \leqslant 0$. We have

$$p(t)Z''_H(t) + (p'(t) + r(t))\, Z'_H(t) \leq 0, \quad t \geq t_2 \tag{16}$$

and from (H_1) it follows that $P'(t) = P(t)\left(\frac{p'(t)+r(t)}{p(t)}\right)$, $P(t) > 0$ and $P'(t) \geqslant 0$ for $t \geqslant t_2$. Multiply both sides of this inequality by $\frac{P(t)}{p(t)}$, we obtain

$$P(t)Z''_H(t) + P'(t)Z'_H(t) = (P(t)Z'_H(t))' \leqslant 0. \tag{17}$$

From (17) we have $P(t)Z'_H(t) \leq P(T)Z'_H(T) \leqslant 0,\ t \geqslant T$. Thus

$$Z_H(t) \leqslant Z_H(T) + P(T)Z'_H(T)\int_T^t \frac{ds}{P(s)} \quad \text{for} \quad t \geqslant T.$$

Again, from (H_1) we have $\lim\limits_{t\to\infty} Z_H(t) = -\infty$ which contradicts the fact that $Z_H(t) > 0$ for $t > 0$. Hence $Z'_H(t) > 0$ and since $\rho(t) \leqslant t$ for $t \geqslant t_1$, we have

$$\begin{aligned} V_H(t) =&\ Z_H(t) - g(t)V_H(\rho(t)) \geqslant Z_H(t) - g(t)Z_H(\rho(t)) \\ &\geqslant Z_H(t)\,(1 - g(t)) \end{aligned}$$

and

$$V_H(\tau(t,\zeta)) \geqslant\ g_0 Z_H(\tau(t,\zeta)).$$

Therefore from (15), we have

$$(p(t)Z'_H(t))' + r(t)Z'_H(t) + g_0\int_c^d Q(t,\zeta)Z_H(\tau(t,\zeta))d\eta(\zeta) \leqslant 0, \quad t \geq t_0$$

From (H_2) and $Z'_H > 0$, we have

$$Z_H(\tau(t,\zeta)) \geqslant Z_H(\tau(t,c)) > 0, \quad \zeta \in [c,d] \quad \text{and} \quad \sigma(t) \leqslant \tau(t,c) \leqslant t.$$

Thus, $Z_H(\sigma(t)) \leqslant Z_H(\tau(t,c))$ and therefore

$$(p(t)Z'_H(t))' + r(t)Z'_H(t) + G(t)Z_H(\sigma(t)) \leqslant 0, \quad t \geq t_1. \tag{18}$$

For $t \geqslant t_0$, $t = t_j,\ j = 1, 2, \cdots$, multiplying both sides of inequality (4) by $1/|\Omega|$ and integrating with respect to x over the domain Ω, we obtain

$$\alpha_j^* \leqslant \frac{V_H(t_j^+)}{V_H(t_j)} \leqslant \alpha_j, \qquad \beta_j^* \leqslant \frac{V'_H(t_j^+)}{V'_H(t_j)} \leqslant \beta_j.$$

Since $Z_H(t) = V_H(t) + g(t)V_H(\rho(t))$, we have that

$$\alpha_j^* \leqslant \frac{Z_H(t_j^+)}{Z_H(t_j)} \leqslant \alpha_j, \qquad \beta_j^* \leqslant \frac{Z'_H(t_j^+)}{Z'_H(t_j)} \leqslant \beta_j. \tag{19}$$

Therefore (18) and (19) show that $Z_H(t) > 0$ is a positive solution of the impulsive differential inequality (10). This is a contradiction.

Suppose, now, that $u_H(x,t) < 0$ is a negative solution of the impulsive partial differential inequality (5) satisfying the boundary condition (2), $(x,t) \in \Omega \times [t_0, +\infty)$, $t_0 > 0$. Using the above procedure, easily, we can reach a contradiction. The proof is complete. □

Theorem 3.4. *Assume that if there exist a function* $\Phi(t) \in C^1(\mathbb{R}_+, (0, +\infty))$ *which is nondecreasing with respect to* t*, such that*

$$\int_{t_0}^{+\infty} \prod_{t_0 \leqslant t_j < s} \left(\frac{\beta_j}{\alpha_j^*}\right)^{-1} \left[\Phi(s)G(s) - \frac{D^2(s)}{4E(s)}\right] ds = +\infty, \tag{20}$$

where

$$D(t) = \frac{\Phi'(t)}{\Phi(t)} - \frac{r(t)}{p(t)} \quad \text{and} \quad E(t) = \frac{\sigma'(t)}{\Phi(\sigma(t))p(\sigma(t))},$$

then every solution of (1)-(2) *is* H*-oscillatory in* $\Omega \times \mathbb{R}_+$.

Proof. We show that the inequality (10) has no eventually positive solution if the conditions of Theorem 3.3 hold. Suppose that $Z_H(t)$ is an eventually positive solution of the inequality (10). Then there exists a number $t_1 > t_0$ such that $Z_H(\sigma(t)) > 0$ for $t > t_1$. Thus we have

$$(p(t)Z'_H(t))' + r(t)Z'_H(t) + G(t)Z_H(\sigma(t)) \leqslant 0. \tag{21}$$

Define

$$W(t) := \Phi(t)\frac{p(t)Z'_H(t)}{Z_H(\sigma(t))},$$

then $W(t) \geqslant 0$ and

$$W'(t) \leqslant \left(\frac{\Phi'(t)}{\Phi(t)} - \frac{r(t)}{p(t)}\right) W(t) - \Phi(t)G(t) - \frac{W^2(t)}{\Phi(\sigma(t))}\frac{\sigma'(t)}{p(\sigma(t))}.$$

Thus $W'(t) \leqslant D(t)W(t) - G(t)\Phi(t) - W^2(t)E(t)$ and $W(t_j^+) \leqslant \dfrac{\beta_j}{\alpha_j^*}W(t_j)$. Define

$$A(t) = \prod_{t_0 \leqslant t_j < t} \left(\tfrac{\beta_j}{\alpha_j^*}\right)^{-1} W(t).$$

It is clear that $W(t)$ is continuous in each interval $(t_j, t_{k+1}]$, and since $W(t_j^+) \leq \frac{\beta_j}{\alpha_j^*} W(t_j)$, it follows that

$$A(t_j^+) = \prod_{t_0 \leqslant t_i \leqslant t_j} \left(\frac{\beta_j}{\alpha_j^*}\right)^{-1} W(t_j^+) \leqslant \prod_{t_0 \leqslant t_i < t_j} \left(\frac{\beta_j}{\alpha_j^*}\right)^{-1} W(t_j) = A(t_j)$$

and for all $t \geqslant t_0$,

$$A(t_j^-) = \prod_{t_0 \leqslant t_i \leqslant t_{j-1}} \left(\frac{\beta_j}{\alpha_j^*}\right)^{-1} W(t_j^-) \leq \prod_{t_0 \leqslant t_i < t_j} \left(\frac{\beta_j}{\alpha_j^*}\right)^{-1} W(t_j) = A(t_j)$$

which implies that $A(t)$ is continuous on $[t_0, +\infty)$.

$$\begin{aligned} & A'(t) + \prod_{t_0 \leqslant t_j < t} \left(\frac{\beta_j}{\alpha_j^*}\right) A^2(t)E(t) + \prod_{t_0 \leqslant t_j < t} \left(\frac{\beta_j}{\alpha_j^*}\right)^{-1} G(t)\Phi(t) - A(t)D(t) \\ &= \prod_{t_0 \leqslant t_j < t} \left(\frac{\beta_j}{\alpha_j^*}\right)^{-1} \left[W'(t) + W^2(t)E(t) - W(t)D(t) + G(t)\Phi(t)\right] \leqslant 0, \end{aligned}$$

that is

$$A'(t) \leqslant - \prod_{t_0 \leqslant t_j < t} \left(\frac{\beta_j}{\alpha_j^*}\right) E(t)A^2(t) + A(t)D(t) - \prod_{t_0 \leqslant t_j < t} \left(\frac{\beta_j}{\alpha_j^*}\right)^{-1} G(t)\Phi(t). \quad (22)$$

Taking

$$X(t) = \left(\prod_{t_0 \leqslant t_j < t} \left(\frac{\beta_j}{\alpha_j^*}\right) E(t)\right)^{\frac{1}{2}} A(t) \quad \text{and} \quad Y(t) = \frac{D(t)}{2} \left(\prod_{t_0 \leq t_j < t} \left(\frac{\beta_j}{\alpha_j^*}\right)^{-1} \frac{1}{E(t)}\right)^{\frac{1}{2}},$$

using Lemma 2.3, we have

$$D(t)A(t) - \prod_{t_0 \leqslant t_j < t} \left(\frac{\beta_j}{\alpha_j^*}\right) E(t)A^2(t) \leqslant \frac{D^2(t)}{4E(t)} \prod_{t_0 \leqslant t_j < t} \left(\frac{\beta_j}{\alpha_j^*}\right)^{-1}.$$

Thus

$$A'(t) \leqslant - \prod_{t_0 \leqslant t_j < t} \left(\frac{\beta_j}{\alpha_j^*}\right)^{-1} \left[G(t)\Phi(t) - \frac{D^2(t)}{4E(t)}\right].$$

Integrating both sides from t_0 to t, we have

$$A(t) \leqslant A(t_0) - \int_{t_0}^{t} \prod_{t_0 \leqslant t_j < s} \left(\frac{\beta_j}{\alpha_j^*}\right)^{-1} \left[G(s)\Phi(s) - \frac{D^2(s)}{4E(s)}\right] ds.$$

Letting $t \to \infty$ and using (20) we have $\lim_{t\to\infty} A(t) = -\infty$, which leads to a contradiction with $A(t) \geqslant 0$ and completes the proof. $\square$

Theorem 3.5. *Assume that there exist functions* Φ *and* $\phi \in C^1(\mathbb{R}_+, (0, +\infty))$ *where* Φ *is nondecreasing, and functions* $b, B \in C^1(\mathbb{D}, \mathbb{R})$*, where* $\mathbb{D} = \{(t,s)\colon t \geqslant s \geqslant t_0 > 0\}$ *such that*

($\mathbf{H_7}$) $B(t,t) = 0$ *and* $B(t,s) > 0$ *for all* $t > s \geqslant t_0$,

($\mathbf{H_8}$) $\dfrac{\partial B(t,s)}{\partial t} \geqslant 0$ *and* $\dfrac{\partial B(t,s)}{\partial s} \leqslant 0$,

($\mathbf{H_9}$) $-\dfrac{\partial B(t,s)}{\partial s} = b(t,s)\sqrt{B(t,s)}$. *If*

$$\limsup_{t\to+\infty} \frac{1}{B(t,t_0)} \int_{t_0}^{t} \prod_{t_0 \leqslant t_j < s} \left(\frac{\beta_j}{\alpha_j^*}\right)^{-1} \Bigg(G(s)\Phi(s)B(t,s)\phi(s) - \frac{1}{4} \frac{[b(t,s)\phi(s) - \phi'(s)\sqrt{B(t,s)} - D(s)\phi(s)\sqrt{B(t,s)}]^2}{E(s)\phi(s)} \Bigg) ds = +\infty, \quad (23)$$

then every solution of (1)-(2) *is* H*-oscillatory in* $\Omega \times \mathbb{R}_+$.

Proof. Let $Z_H(t)$ is an eventually positive solution of (10). Proceeding as in the proof of Theorem 3.4 we obtain

$$A'(t) \leqslant -\prod_{t_0 \leqslant t_j < t} \left(\frac{\beta_j}{\alpha_j^*}\right) E(t)A^2(t) + A(t)D(t) - \prod_{t_0 \leqslant t_j < t} \left(\frac{\beta_j}{\alpha_j^*}\right)^{-1} G(t)\Phi(t)$$

multiplying the above inequality by $B(t,s)\phi(s)$ for $t \geqslant s \geqslant T$, and integrating from T to t, we have

$$\begin{aligned} \int_T^t A'(s)B(t,s)\phi(s)ds \leqslant\; & -\int_T^t \prod_{t_0 \leqslant t_j < s} \left(\frac{\beta_j}{\alpha_j^*}\right) E(s)A^2(s)B(t,s)\phi(s)ds \\ & + \int_T^t A(s)D(s)B(t,s)\phi(s)ds \\ & - \int_T^t \prod_{t_0 \leqslant t_j < s} \left(\frac{\beta_j}{\alpha_j^*}\right)^{-1} G(s)\Phi(s)B(t,s)\phi(s)ds. \end{aligned}$$

Thus we have

$$\begin{aligned} \int_T^t \prod_{t_0 \leqslant t_j < s} \left(\frac{\beta_j}{\alpha_j^*}\right)^{-1} G(s)\Phi(s)B(t,s)\phi(s)ds \leqslant\; & A(T)B(t,T)\phi(T) \\ & - \int_T^t \left[-\frac{\partial B(t,s)}{\partial s}\phi(s) - B(t,s)\phi'(s) - D(s)B(t,s)\phi(s) \right] A(s)ds \\ & - \int_T^t \prod_{t_0 \leqslant t_j < s} \left(\frac{\beta_j}{\alpha_j^*}\right) E(s)A^2(s)B(t,s)\phi(s)ds. \end{aligned}$$

Therefore

$$\int_T^t \prod_{t_0\leqslant t_j<s}\left(\frac{\beta_j}{\alpha_j^*}\right)^{-1} G(s)\Phi(s)B(t,s)\phi(s)ds$$
$$-\frac{1}{4}\int_T^t \prod_{t_0\leqslant t_j<s}\left(\frac{\beta_j}{\alpha_j^*}\right)^{-1}\frac{\left[b(t,s)\phi(s)-\phi'(s)\sqrt{B(t,s)}-D(s)\phi(s)\sqrt{B(t,s)}\right]^2}{E(s)\phi(s)}$$
$$\leqslant A(T)B(t,T)\phi(T). \tag{24}$$

From (24) for $t\geqslant T\geqslant t_0$, we have

$$\frac{1}{B(t,t_0)}\int_{t_0}^t \prod_{t_0\leqslant t_j<s}\left(\frac{\beta_j}{\alpha_j^*}\right)^{-1}\Bigg[G(s)\Phi(s)B(t,s)\phi(s)$$
$$-\frac{1}{4}\frac{\left[b(t,s)\phi(s)-\phi'(s)\sqrt{B(t,s)}-D(s)\phi(s)\sqrt{B(t,s)}\right]^2}{E(s)\phi(s)}\Bigg]ds$$
$$=\frac{1}{B(t,t_0)}\left[\int_{t_0}^T+\int_T^t\right]\Bigg\{\prod_{t_0\leq t_j<s}\left(\frac{\beta_j}{\alpha_j^*}\right)^{-1}\Bigg(G(s)\Phi(s)B(t,s)\phi(s)$$
$$-\frac{1}{4}\frac{\left[b(t,s)\phi(s)-\phi'(s)\sqrt{B(t,s)}-D(s)\phi(s)\sqrt{B(t,s)}\right]^2}{E(s)\phi(s)}\Bigg)\Bigg\}ds$$
$$\leqslant\int_{t_0}^T \prod_{t_0\leqslant t_j<s}\left(\frac{\beta_j}{\alpha_j^*}\right)^{-1} G(s)\Phi(s)\phi(s)ds+\phi(T)A(T).$$

Letting $t\to+\infty$, we have

$$\limsup_{t\to+\infty}\frac{1}{B(t,t_0)}\int_{t_0}^t \prod_{t_0\leqslant t_j<s}\left(\frac{\beta_j}{\alpha_j^*}\right)^{-1}\Bigg(G(s)\Phi(s)B(t,s)\phi(s)$$
$$-\frac{1}{4}\frac{\left[b(t,s)\phi(s)-\phi'(s)\sqrt{B(t,s)}-D(s)\phi(s)\sqrt{B(t,s)}\right]^2}{E(s)\phi(s)}\Bigg)ds$$
$$\leqslant\int_{t_0}^T \prod_{t_0\leqslant t_j<s}\left(\frac{\beta_j}{\alpha_j^*}\right)^{-1} G(s)\Phi(s)\phi(s)ds+\phi(T)A(T)$$
$$<+\infty,$$

which leads to a contradiction with (23) and completes the proof. □

Choosing $\phi(s)=\Phi(s)\equiv 1$, in Theorem 3.5, we establish the following result.

Corollary 3.6. *Assume that the conditions of Theorem 3.5 hold and*

$$\limsup_{t\to+\infty}\frac{1}{B(t,t_0)}\int_{t_0}^{t}\prod_{t_0\leqslant t_j<s}\left(\frac{\beta_j}{\alpha_j^*}\right)^{-1}\left(G(s)B(t,s)-\frac{1}{4}\frac{\left[b(t,s)-D(s)\sqrt{B(t,s)}\right]^2}{E(s)}\right)ds=+\infty,$$

then every solution of (1)-(2) *is H-oscillatory in* $\Omega\times\mathbb{R}_+$.

From Theorem 3.5 and Corollary 3.6, we can have several oscillatory criteria by different choices of the weighted function $B(t,s)$. For example, choosing $B(t,s)=(t-s)^{\lambda-1}$, $t\geqslant s\geqslant t_0$, in which $\lambda>2$ is an integer, then $b(t,s)=(\lambda-1)(t-s)^{(\lambda-3)/2}$, $t\geqslant s\geqslant t_0$. Corollary 3.6 leads to the following result.

Corollary 3.7. *If $\lambda>2$ is an integer such that*

$$\limsup_{t\to+\infty}\frac{1}{(t-t_0)^{\lambda-1}}\int_{t_0}^{t}\prod_{t_0\leqslant t_j<s}\left(\frac{\beta_j}{\alpha_j^*}\right)^{-1}(t-s)^{\lambda-1}\left(G(s)-\frac{1}{4E(s)}\times\right.$$
$$\left.\left[D^2(s)-\frac{2(\lambda-1)D(s)}{(t-s)}+\frac{(\lambda-1)^2}{(t-s)^2}\right]\right)ds=+\infty,$$

then every solution of (1)-(2) *is H-oscillatory in* $\Omega\times\mathbb{R}_+$.

Now we consider $B(t,s)=[P(t)-P(s)]^{\lambda}$, $t\geqslant s\geqslant t_0$, where $P(t)=\int_{t_0}^{t}\frac{1}{p(s)}ds$ and $\lim_{t\to+\infty}P(t)=+\infty$, then $b(t,s)=\lambda[P(t)-P(s)]^{\lambda-2/2}$. This leads to the result in the following Corollary.

Corollary 3.8. *If $\lambda>2$ is an integer such that*

$$\limsup_{t\to+\infty}\frac{1}{[P(t)-P(t_0)]^{\lambda}}\int_{t_0}^{t}\prod_{t_0\leqslant t_j<s}\left(\frac{\beta_j}{\alpha_j^*}\right)^{-1}[P(t)-P(s)]^{\lambda}\left(G(s)-\frac{1}{4E(s)}\times\right.$$
$$\left.\left[D^2(s)-\frac{2\lambda D(s)}{P(t)-P(s)}+\frac{\lambda^2}{[P(t)-P(s)]^2}\right]\right)ds=+\infty,$$

then every solution of (1)-(2) *is H-oscillatory in* $\Omega\times\mathbb{R}_+$.

4. H-Oscillation of (1)-(3)

In this section, we establish sufficient conditions for the H-oscillation of all solutions of (1)-(3).

Theorem 4.1. *Let H be a fixed unit vector in $\mathbb{R}^M$. If the impulsive differential inequality*

$$\begin{cases} \Big(p(t)\tilde{Z}_H'(t)\Big)' + r(t)\tilde{Z}_H'(t) + G(t)\tilde{Z}_H(\sigma(t)) \leqslant 0, \quad t \neq t_j \\ \alpha_j^* \leqslant \dfrac{\tilde{Z}_H(t_j^+)}{\tilde{Z}_H(t_j)} \leqslant \alpha_j, \quad \beta_j^* \leqslant \dfrac{\tilde{Z}_H'(t_j^+)}{\tilde{Z}_H'(t_j)} \leqslant \beta_j \quad j = 1, 2, \cdots, \end{cases} \tag{25}$$

has no eventually positive solution and the impulsive differential inequality

$$\begin{cases} \Big(p(t)\tilde{Z}_H'(t)\Big)' + r(t)\tilde{Z}_H'(t) + G(t)\tilde{Z}_H(\sigma(t)) \geqslant 0, \quad t \neq t_j \\ \alpha_j^* \geqslant \dfrac{\tilde{Z}_H(t_j^+)}{\tilde{Z}_H(t_j)} \geqslant \alpha_j, \quad \beta_j^* \geqslant \dfrac{\tilde{Z}_H'(t_j^+)}{\tilde{Z}_H'(t_j)} \geqslant \beta_j \quad j = 1, 2, \cdots, \end{cases} \tag{26}$$

has no eventually negative solution, then every solution $U(x,t)$ of (1)-(3) *is H-oscillatory in $\Omega \times \mathbb{R}_+$.*

Proof. Suppose to the contrary that there exists a solution $U(x,t)$ of (1)-(3) which is not H-oscillatory in G. Without loss of generality, we may assume that $u_H(x,t) > 0$ in $\Omega \times [t_0, +\infty)$, for some $t_0 > 0$.

For $t \geqslant t_0$ and $t \neq t_j$ for $j = 1, 2, \cdots$, we multiply both sides of inequality (4) by $K_\varphi \varphi(x)$ and integrate with respect to x over the domain Ω to attain

$$\begin{cases} \dfrac{\partial}{\partial t}\left(p(t)\dfrac{\partial}{\partial t}\left(\displaystyle\int_\Omega K_\varphi \varphi(x) u_H(x,t)dx + \int_\Omega g(t) K_\varphi \varphi(x) u_H(x,\rho(t))dx\right)\right) \\ +r(t)\dfrac{\partial}{\partial t}\left(\displaystyle\int_\Omega K_\varphi \varphi(x) u_H(x,t)dx + \int_\Omega g(t) K_\varphi \varphi(x) u_H(x,\rho(t))dx\right) \\ +\displaystyle\int_\Omega \int_c^d K_\varphi \varphi(x) Q(t,\zeta) u_H(x,\tau(t,\zeta)) d\eta(\zeta) dx - a(t)\int_\Omega K_\varphi \varphi(x) \Delta u_H(x,t) dx \\ -\displaystyle\int_\Omega \int_c^d a(t,\zeta) K_\varphi \varphi(x) \Delta u_H(x,\theta(t,\zeta)) d\eta(\zeta) dx \leqslant \int_\Omega K_\varphi \varphi(x) f_H(x,t) dx, \quad t \neq t_j. \end{cases} \tag{27}$$

Using Green's formula and boundary condition $(3')$, we have that

$$\begin{aligned} \int_\Omega K_\varphi \varphi(x) \Delta u_H(x,t) dx = {} & K_\varphi \int_{\partial\Omega} \left[\varphi(x)\frac{\partial u_H(x,t)}{\partial \gamma} - u_H(x,t)\frac{\partial \varphi(x)}{\partial \gamma}\right] dS \\ & + K_\varphi \int_\Omega u_H(x,t) \Delta \varphi(x) dx = 0 - \lambda_0 \tilde{V}_H(t) \leqslant 0, \end{aligned} \tag{28}$$

and

$$\begin{aligned}\int_\Omega K_\varphi\varphi(x)\Delta u_H(x,\theta(t,\zeta))dx = &\ K_\varphi\int_{\partial\Omega}\left[\varphi(x)\frac{\partial u_H(x,\theta(t,\zeta))}{\partial\gamma} - u_H(x,\theta(t,\zeta))\frac{\partial\varphi(x)}{\partial\gamma}\right]dS \\ &+ K_\varphi\int_\Omega u_H(x,\theta(t,\zeta))\Delta\varphi(x)dx \\ =&\ 0 - \lambda_0\tilde{V}_H(\theta(t,\zeta)) \leqslant 0 \end{aligned} \tag{29}$$

where dS is the surface component on $\partial\Omega$. Furthermore, by (H_4), $\int_\Omega f_H(x,t)dx \leqslant 0$. In view of (27)-(29), we get

$$\begin{aligned}&\frac{d}{dt}\left[p(t)\frac{d}{dt}\left(\tilde{V}_H(t)+g(t)\tilde{V}_H(\rho(t))\right)\right] + r(t)\frac{d}{dt}\left(\tilde{V}_H(t)+g(t)\tilde{V}_H(\rho(t))\right)\\ &+\int_c^d Q(t,\zeta)\tilde{V}_H(\tau(t,\zeta))d\eta(\zeta) \leqslant 0, \quad t\geq t_0.\end{aligned}$$

Setting $\tilde{Z}_H(t) = \tilde{V}_H(t) + g(t)\tilde{V}_H(\rho(t))$, we have

$$\left(p(t)\tilde{Z}'_H(t)\right)' + r(t)\tilde{Z}'_H(t) + \int_c^d Q(t,\zeta)\tilde{V}_H(\tau(t,\zeta))d\eta(\zeta) \leqslant 0.$$

The rest of the proof is similar to that of Theorem 3.3 and hence the details are omitted. The proof is complete. □

Theorem 4.2. *Let the conditions of Theorem 3.4 hold, then every solution of (1)-(3) is H-oscillatory in $\Omega\times\mathbb{R}_+$.*

Theorem 4.3. *Let the conditions of Theorem 3.5 hold, then every solution of (1)-(3) is H-oscillatory in $\Omega\times\mathbb{R}_+$.*

Corollary 4.4. *Let the conditions of Corollary 3.6 hold, then every solution of (1)-(3) is H-oscillatory in $\Omega\times\mathbb{R}_+$.*

Corollary 4.5. *Let the conditions of Corollary 3.7 hold, then every solution of (1)-(3) is H-oscillatory in $\Omega\times\mathbb{R}_+$.*

Corollary 4.6. *Let the conditions of Corollary 3.8 hold, then every solution of (1)-(3) is H-oscillatory in $\Omega\times\mathbb{R}_+$.*

5. Example

In this section we provide an example to point up our result.

Example 5.1. *Consider the following impulsive partial differential equations*

$$\begin{cases} \dfrac{\partial}{\partial t}\Big(3\dfrac{\partial}{\partial t}\Big(U(x,t)+\dfrac{1}{5}U(x,t-\pi)\Big)\Big)+\Big(-\dfrac{3}{4}\Big)\dfrac{\partial}{\partial t}\Big(U(x,t)+\dfrac{1}{5}U(x,t-\pi)\Big) \\ +\dfrac{4}{5}\displaystyle\int_{\pi/4}^{\pi/2}U(x,t-2\zeta)d\zeta=5\Delta U(x,t)+\dfrac{22}{5}\displaystyle\int_{\pi/4}^{\pi/2}\Delta U(x,t-2\zeta)d\zeta+F(x,t), \quad t\neq t_j, \\ U(x,t_j^+)=\dfrac{j+1}{j}U(x,t_j), \\ \dfrac{\partial}{\partial t}U(x,t_j^+)=\dfrac{\partial}{\partial t}U(x,t_j), \quad j=1,2,\cdots, \end{cases} \tag{30}$$

for $x\in(0,\pi),\ t\in\mathbb{R}_+$, *with the boundary condition*

$$U(0,t)=U(\pi,t)=\begin{pmatrix}0\\0\end{pmatrix}, \quad t\neq t_j,\ j=1,2,\cdots. \tag{31}$$

Here $\Omega=(0,\pi)$, $N=1$, $M=2$, $\alpha_j=\alpha_j^*=\dfrac{j+1}{j}$, $\beta_j=\beta_j^*=1$, $p(t)=3$, $g(t)=\dfrac{1}{5}$, $\rho(t)=t-\pi$, $r(t)=-\dfrac{3}{4}$, $\tau(t,\zeta)=\theta(t,\zeta)=t-2\zeta$, $\eta(\zeta)=\zeta$, $Q(t,\zeta)=\dfrac{4}{5}$, $a(t)=5$, $a(t,\zeta)=\dfrac{22}{5}$, $[c,d]=[\pi/4,\pi/2]$,

$$F(x,t)=\begin{pmatrix} -\dfrac{16}{5}\sin x\ \cos t \\ \sin x\ e^t\Big(\dfrac{29}{4}+\dfrac{13}{5}e^{-\pi/2}-\dfrac{43}{20}e^{-\pi}\Big)\end{pmatrix}.$$

Let $H=e_1=\begin{pmatrix}1\\0\end{pmatrix}$, *then we have* $f_H(x,t)=f_{e_1}(x,t)=-\dfrac{16}{5}\sin x\cos t$ *and*

$$\int_\Omega f_{e_1}(x,t)dx=-\frac{16}{5}\int_\Omega \sin x\cos t dx=-\frac{32}{5}\cos t\le 0, \quad \frac{\pi}{2}\le t\le\frac{3\pi}{2}.$$

Take $\lambda=3$, $\sigma(t)=t$, $\sigma'(t)=1$, $\Phi(t)=1$. *Since* $t_0=1$, $t_j=2^j$, $E(s)=\dfrac{1}{3}$, $D(s)=\dfrac{1}{4}$, $G(s)=\dfrac{4\pi}{25}$. *Then hypotheses* $(H_1)-(H_6)$ *hold, and further*

$$\begin{aligned}\lim_{t\to+\infty}\int_{t_0}^{t}\prod_{t_0\leqslant t_j<s}\frac{\beta_j^*}{\alpha_j}ds &= \int_1^{+\infty}\prod_{1<t_j<s}\frac{j}{j+1}ds \\ &= \int_1^{t_1}\prod_{1<t_j<s}\frac{j}{j+1}ds+\int_{t_1^+}^{t_2}\prod_{1<t_j<s}\frac{j}{j+1}ds+\int_{t_2^+}^{t_3}\prod_{1<t_j<s}\frac{j}{j+1}ds+\cdots \\ &= 1+\frac{1}{2}\times 2+\frac{1}{2}\times\frac{2}{3}\times 2^2+\cdots=\sum_{n=0}^{\infty}\frac{2^n}{n+1}=+\infty.\end{aligned}$$

Thus

$$\limsup_{t\to+\infty}\frac{1}{(t-1)^2}\left\{\int_1^t\prod_{1<t_j<s}\frac{j+1}{j}(t-s)^2\left[\frac{4\pi}{25}-\frac{3}{64}+\frac{3}{4(t-s)}-\frac{3}{(t-s)^2}\right]ds\right\}=+\infty.$$

Therefore all the conditions of Corollary 4.5 are satisfied and hence every solution $U(x,t)$ *of equation* (30)-(31) *is* e_1*-oscillatory in* G*. In fact*

$$U(x,t)=\begin{pmatrix}\sin x\ \sin t\\ \\ \sin x\ e^t\end{pmatrix},$$

is one such solution of (30)-(31). *We note the above solution* $U(x,t)$ *is not* e_2*-oscillatory in* G*, where* $e_2=\begin{pmatrix}0\\1\end{pmatrix}$.

6. Conclusion:

In this paper, we have established some new oscillation criteria for impulsive vector partial differential equations with damping term. We have derived sufficient conditions for the H-oscillation of solutions, using impulsive differential inequalities and average technique with two different boundary conditions. The example we have used is illustrative of the application of Corollary 4.5. The present results complement and extend those established for problems without impulses.

References

[1] M. Benchohra, J. Henderson and S. Ntouyas, Impulsive Differential Equations and Inclusions, *Hindawi Publishing Corporation, New York,* 2006.

[2] Ju.I. Domšlak, On the oscillation of solutions of vector differential equations, *Soviet Math. Dokl.,* **11**(1970), 839-841.

[3] Yu.I. Domšlak, Oscillatory properties of solutions of vector differential equations, *Differ. Uravn.,* **7**(1971), 961-969.

[4] L.H. Erbe, H.I. Freedman, X.Z. Liu and J.H. Wu, Comparison principles for impulsive parabolic equations with application to models of single species growth, *J. Austral. Math. Soc. Ser(B).,* **32**(4)(1991), 382-400.

[5] Z. Gao and Z. Teng, Oscillation criteria of impulsive neutral parabolic equations with nonlinear diffusion coefficient, *Int. J. Nonlinear Sci.,* **11**(2)(2011), 168-172.

[6] K. Gopalsamy and B.G. Zhang, On delay differential equations with impulses, *J. Math. Anal. Appl.,* **139**(1989), 110-122.

[7] G.H. Hardy, J.E. Littlewood and G. Pólya, Inequalities, *Cambridge University Press, Cambridge, UK,* 1988.

[8] K. Kreith, A nonselfadjoint dynamical system, *Proc. Edinburgh Math. Soc.,* **19**(2)(1974), 77-87.

[9] G.S. Ladde, V. Lakshmikantham and B.G. Zhang, Oscillation Theory of Differential Equations with Deviating Arguments, *Marcel Dekker, Inc, New York,* 1987.

[10] V. Lakshmikantham, D.D. Bainov and P.S. Simeonov, Theory of Impulsive Differential Equations, *World Scientific Publishers, Singapore,* 1989.

[11] W.N. Li and M. Han, Oscillation of solutions for certain impulsive vector parabolic differential equations with delays, *J. Math. Anal. Appl.,* **326**(1)(2007), 363-371.

[12] W.N. Li, M. Han and F.W. Meng, H-oscillation of solutions of certain vector hyperbolic differential equations with deviating arguments, *Appl. Math. Comput.,* **158**(2004), 637-653.

[13] G.J. Liu and C.Y. Wang, Forced oscillation of neutral impulsive parabolic partial differential equations with continuous distributed deviating arguments, *Open Access Library Journal,* **1**(9)(2014), 1-8.

[14] J.W. Luo, Oscillation of hyperbolic partial differential equations with impulses, *Appl. Math. Comput.,* **133**(2-3)(2002), 309-318.

[15] Q.X. Ma and A.P. Liu, Oscillation criteria of neutral type impulsive hyperbolic equations, *Acta Math. Sci.,* **34B**(6)(2014), 1845-1853.

[16] Q.X. Ma, L.L. Zhang and A.P. Liu, Oscillation of nonlinear impulsive hyperbolic equations of neutral type, *Applied Mechanics and Materials,* **275-277**(2013), 848-851.

[17] V.D. Mil'Man and A.D. Myshkis, On the stability of motion in the presence of impulse, *Sib. Math. J.,* **1**(2)(1960), 233-237.

[18] E. Minchev and N. Yoshida, Oscillation of vector differential equations of hyperbolic type with functional arguments, *Math. J. Toyama Univ.,* **26**(2003), 75-84.

[19] E. Minchev and N. Yoshida, Oscillation of solutions of vector differential equations of parabolic type with functional arguments, *J. Comput. Appl. Math.,* **151**(1)(2003), 107-117.

[20] E.S. Noussair and C.A. Swanson, Oscillation theorems for vector differential equations, *Util. Math.,* **1**(1972), 97-109.

[21] E.S. Noussair and C.A. Swanson, Oscillation of nonlinear vector differential equations, *Ann. Math. Pura Appl.,* **109**(1)(1976), 305-315.

[22] P. Prakash and S. Harikrishnan, Oscillation of solutions of impulsive vector hyperbolic differential equations with delays, *Applicable Analysis,* **91**(3)(2012), 459-473.

[23] V. Sadhasivam and J. Kavitha, On the Oscillation of solutions of fractional vector partial differential equations with deviating arguments, *American Review of Mathematics and Statistics*, (accepted).

[24] V. Sadhasivam, J. Kavitha and T. Raja, Forced oscillation of nonlinear impulsive hyperbolic partial differential equation with several delays, *Journal of Applied Mathematics and Physics,* **3**(2015), 1491-1505.

[25] V. Sadhasivam, J. Kavitha and T. Raja, Forced oscillation of impulsive neutral hyperbolic differential equations, *International Journal of Applied Engineering Research,* **11**(1)(2016), 58-63.

[26] V. Sadhasivam, T. Raja and T. Kalaimani, Oscillation of nonlinear impulsive neutral functional hyperbolic equations with damping, *International Journal of Pure and Applied Mathematics*, **106**(8)(2016), 187-197.

[27] A.M. Samoilenko and N.A. Persestyuk, Impulsive Differential Equations, *World Scientific, Singapore,* 1995.

[28] V.S. Vladimirov, Equations of Mathematics Physics, Nauka, Moscow, 1981.

[29] J.H. Wu, Theory and Applications of Partial Functional Differential Equations, *Springer-Verlag, New York,* 1996.

[30] J. Yang, A.P. Liu and G. Liu, Oscillation of solutions to neutral nonlinear impulsive hyperbolic equations with several delays, *Electron. J. Differential Equations,* **2013**(27)(2013), 1-10.

[31] W. Yang, J. Liu, S. Zheng and L. Li, Oscillation criteria of solutions for a new class of impulsive parabolic equations with delay, *Int. J. Advanced Mechatronic Systems,* **5**(1)(2013), 3-11.

[32] N. Yoshida, Oscillation Theory of Partial Differential Equations, *World Scientific, Singapore,* 2008.