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# Sum of Finite and Infinite Series Derived by Generalized Mixed Difference Equation

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Abstract. In this work, by introducing the generalized mixed $q$-$\ell$ difference equation with the product of generalized difference operator $\Delta_\ell$ and the q-difference operator $\Delta_q$, we derive mixed multi-summation formula and mixed higher order summation formula for certain functions. Suitable numerical examples verified by MATLAB are also provided.

**Key words:** Generalized mixed difference operator, Polynomial factorial and Summation solution.

**AMS Subject classification:** 39A10, 39A11, 39A13, 39A70, 49M.

## 1. Introduction

The modern theory of differential or integral calculus began in the $17^{th}$ century with the works of Newton and Leibnitz [1]. In 1984, Jerzy Popenda [2] introduced a particular type of difference operator $\Delta_\alpha$ defined on $u(k)$ as $\Delta_\alpha u(k) = u(k+1) - \alpha u(k)$. In 1989, K.S.Miller and Ross [3] introduced the discrete analogue of the Riemann-Liouville fractional derivative and proved some properties of the fractional derivative operator. The general fractional h-difference Riemann-Liouville operator and its inverse $\Delta_h^{-\nu} f(t)$ were mentioned in ( [4], [5]). As application of $\Delta_h^{-\nu}$, by taking $\nu = m$ (positive integer) and $h = \ell$, the sum of $m^{th}$ partial sums on $n^{th}$ powers of arithmetic, arithmetic-geometric

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progressions and products of n consecutive terms of arithmetic progression have been derived using $\Delta_\ell^{-m}u(k)$, where $\Delta_\ell u(k) = u(k+\ell) - u(k)$ [6].

In 2011, M.Maria Susai Manuel, et al. [7] have extended the definition of $\Delta_\alpha$ to $\underset{\alpha(\ell)}{\Delta}$ which is defined as $\underset{\alpha(\ell)}{\Delta} v(k) = v(k+\ell) - \alpha v(k)$ for the real valued function $v(k)$, $\ell > 0$. In [8], the authors have used the generalized $\alpha$-difference equation $v(k+\ell) - \alpha v(k) = u(k)$, $k \in [0,\infty)$, $\ell > 0$ is fixed, and obtained a summation solution in the form

$$v(k) = \underset{\alpha(\ell)}{\Delta^{-1}}\ u(k) - \alpha^{[\frac{k}{\ell}]} \underset{\alpha(\ell)}{\Delta^{-1}}\ u(\hat{\ell}(k)) = \sum_{r=1}^{[k/\ell]} \alpha^{r-1} u(k - r\ell),\ \hat{\ell}(k) = k - [k/\ell]\ell.$$

In 2014, G.Britto Antony Xavier et al. [9] introduced a $q$-difference operator $\Delta_q$, which is defined as

$$\Delta_q v(k) = v(qk) - v(k), \quad q \neq 1, \tag{1}$$

and obtained a summation solution of the $q$-difference equation $\Delta_q^t v(k) = u(k)$, $k \in (-\infty,\infty)$ and $q \neq 1$, in the form

$$\Delta_q^{-t} u(k)\Big|\Big|_{\frac{k}{q^m}}^{k} = \sum_{(r)_{1\to t}}^{m} u\Big(k \prod_{i=1}^{t} q^{-r_i}\Big).$$

With this background, in this paper, we obtain multi-series solution for the generalized mixed $q$-$\ell$ difference equation

$$\underset{\ell_{1\to n}}{\Delta}\ \underset{q_{1\to t}}{\Delta} v(k) = u(k),\ k \in [0,\infty), \tag{2}$$

where $\underset{\ell_{1\to n}}{\Delta}\ \underset{q_{1\to t}}{\Delta} v(k) = \Delta_{\ell_1}\left(\Delta_{\ell_2}\left(\cdots \Delta_{\ell_n}\left(\Delta_{q_1}\left(\Delta_{q_2}\left(\cdots \Delta_{q_t}\left(v(k)\right)\cdots\right)\right)\right)\cdots\right)\right).$

## 2. Preliminaries

In this section, we present some notations, basic definitions and preliminary results which will be used for the subsequent discussions. Let $u(k)$ be a real valued function defined on $[0,\infty)$, $\hat{\ell}_i(k) = k - [k/\ell_i]\ell_i$, $[k/\ell_i]$ denotes the integer part of $\frac{k}{\ell_i}$ and $m$ is a positive integer. For simplicity, we use the following notations:

(i) $\displaystyle\sum_{(h)_{1\to t}}^{m} = \sum_{h_1=0}^{m_1}\sum_{h_2=0}^{m_2}\cdots\sum_{h_t=0}^{m_t}$; (ii) $\displaystyle\sum_{(r\ell)_{1\to i}}^{[k]} = \sum_{r_1=0}^{[\frac{k}{\ell_1}]}\ \sum_{r_2=0}^{[\frac{k-r_1\ell_1}{\ell_2}]}\cdots \sum_{r_i=0}^{\left[\frac{k-r_1\ell_1-r_2\ell_2-\cdots-r_{i-1}\ell_{i-1}}{\ell_i}\right]}$;

(iii) $\underset{\ell_{1\to n}}{\Delta^{-1}} = \Delta^{-1}_{\ell_1}\Delta^{-1}_{\ell_2}\Delta^{-1}_{\ell_3}\cdots\Delta^{-1}_{\ell_n}$; (iv) $\underset{q_{1\to t}}{\Delta^{-1}} = \Delta^{-1}_{q_1}\Delta^{-1}_{q_2}\Delta^{-1}_{q_3}\cdots\Delta^{-1}_{q_t}$ and

(v) $\underset{\ell_{1\to n}}{\Delta^{-1}}\ \underset{q_{1\to t}}{\Delta^{-1}} u(k)\Big|_{\frac{1}{q_1^{m_1}}\sim\frac{1}{q_t^{m_t}}\sim(\hat{\ell}_1(k)+\sum\limits_{J=2}^{n}\ell_J)}^{q_1\sim q_t\sim(k+\sum\limits_{J=1}^{n}\ell_J)} = \underset{\ell_{1\to n}}{\Delta^{-1}}\ \underset{q_{1\to t}}{\Delta^{-1}} u(k)\Big|_{\frac{k}{q_1^{m_1}}}^{q_1k}\cdots\Big|_{\frac{k}{q_t^{m_t}}}^{q_tk}\Big|_{(\hat{\ell}_1(k)+\sum\limits_{J=2}^{n}\ell_J)}^{(k+\sum\limits_{J=1}^{n}\ell_J)}.$

**Lemma 2.1.** *[10] Let $s_r^n$ and $S_r^n$ be the Stirling numbers of first and second kinds respectively, $n\in N(1)$. If $k_q^{(n)}=\prod\limits_{i=0}^{n-1}(k-iq)$ and $\left(\frac{1}{k}\right)_q^{(n)}=\prod\limits_{i=0}^{n-1}\left(\frac{1}{k}-iq\right)$, $q\neq 0$, then*

$$k_q^{(n)}=\sum_{r=1}^{n}s_r^nq^{n-r}k^r,\ \ k^n=\sum_{r=1}^{n}S_r^nq^{n-r}k_q^{(r)}\ \text{ and }\ \left(\frac{1}{k}\right)_q^{(n)}=\sum_{r=1}^{n}s_r^nq^{n-r}\left(\frac{1}{k}\right)^r. \tag{3}$$

**Lemma 2.2.** *[11] For $k\in(0,\infty)$ and $\ell>0$, we have*

$$\Delta_\ell^{-n}k_\ell^{(m)}=\frac{k_\ell^{(m+n)}}{\ell^n(m+n)^{(n)}} \tag{4}$$

*and*

$$\Delta_\ell^{-1}u(k+\ell)-\Delta_\ell^{-1}u(\hat{\ell}(k))=\sum_{r=0}^{[k/\ell]}u(k-r\ell). \tag{5}$$

## 3. Main Results

The purpose of this section is obtaining the sum of mixed $q$-$\ell$ multi-series by equating summation and closed form solutions of mixed difference equation (2).

**Theorem 3.1.** *[Finite mixed summation formula] Let $q\neq 0$, $\ell>0$, $m\in N(1)$ and $u(k)$ be a real valued function defined on $[0,\infty)$. Then*

$$\sum_{r=0}^{[k/\ell]}\sum_{h=0}^{m}u\Big(\frac{k-r\ell}{q^h}\Big)=\Delta_\ell^{-1}\Delta_q^{-1}u(k)\Big|_{\frac{1}{q^m}\sim\hat{\ell}(k)}^{q\sim(k+\ell)} \tag{6}$$

*is a solution of the mixed difference equation $\Delta_\ell\Delta_q v(k)=u(k)$.*

*Proof.* From (1) and by taking $\Delta_q\ v(k)=u(k)$, we have

$$v(qk)=u(k)+v(k). \tag{7}$$

Replacing $k$ by $k/q$ in (7), we get

$$v(k)=u\Big(\frac{k}{q}\Big)+\ v\Big(\frac{k}{q}\Big). \tag{8}$$

Again replacing $k$ by $k/q^h$, $h=1,2,\cdots,(m-1)$ in (8) repeatedly and substituting the resultant expressions in (7), we arrive

$$u(k)+u\Big(\frac{k}{q}\Big)+u\Big(\frac{k}{q^2}\Big)+\cdots+u\Big(\frac{k}{q^m}\Big)=v(qk)-v\Big(\frac{k}{q^m}\Big),$$

$$\text{i.e.,}\ \sum_{h=0}^{m} u\Big(\frac{k}{q^h}\Big) = \Delta_q^{-1}u(qk) - \Delta_q^{-1}u\Big(\frac{k}{q^m}\Big). \tag{9}$$

For $r = 1, 2, 3, \cdots, [k/\ell]$, replacing $k$ by $k - r\ell$ in (9) and adding all the resultant expressions, we find that

$$\sum_{r=1}^{[k/\ell]}\sum_{h=0}^{m} u\Big(\frac{k-r\ell}{q^h}\Big) = \sum_{r=1}^{[k/\ell]} \Delta_q^{-1}u(q(k-r\ell)) - \sum_{r=1}^{[k/\ell]} \Delta_q^{-1}u\Big(\frac{k-r\ell}{q^m}\Big). \tag{10}$$

Adding (9) and (10) and applying (5), we arrive

$\sum\limits_{r=0}^{[k/\ell]}\sum\limits_{h=0}^{m} u\big(\frac{k-r\ell}{q^h}\big) = \Delta_\ell^{-1}\Delta_q^{-1}\Big\{u(q(k+\ell)) - u(q\hat{\ell}(k)) - \Big(u\big(\frac{k+\ell}{q^m}\big) - u\big(\frac{\hat{\ell}(k)}{q^m}\big)\Big)\Big\}$,

which completes the proof of the Theorem. □

**Theorem 3.2.** *[Multi mixed-summation formula] Let* $k \in \big[\sum_{i=1}^{n}\ell_i, \infty\big)$, $q_i \neq 0$, $\ell_i > 0$ *and* $m$ *is any positive integer. Then equation (2) has a solution of the form*

$$\sum_{i=1}^{n-1}\sum_{(r\ell)_{1\to i}}^{[k]} \Delta^{-1}_{\ell_{i+1\to n}}\Delta^{-1}_{q_{1\to t}} u(k)\Big|_{\frac{1}{q_1^{m_1}}\sim\frac{1}{q_t^{m_t}}}^{q_1\sim q_t\sim\big(\hat{\ell}_{i+1}(k-\sum\limits_{J=1}^{i} r_J\ell_J)+\sum\limits_{J=i+2}^{n}\ell_J\big)}$$

$$+\sum_{(r\ell)_{1\to n}}^{[k]}\sum_{(h)_{1\to t}}^{m} u\left(\frac{k-\sum\limits_{j=1}^{n} r_j\ell_j}{\prod\limits_{p=1}^{t} q_p^{h_p}}\right) = \Delta^{-1}_{\ell_{1\to n}}\Delta^{-1}_{q_{1\to t}} u(k)\Big|_{\frac{1}{q_1^{m_1}}\sim\frac{1}{q_t^{m_t}}\sim\big(\hat{\ell}_1(k)+\sum\limits_{J=2}^{n}\ell_J\big)}^{q_1\sim q_t\sim(k+\sum\limits_{J=1}^{n}\ell_J)}, \tag{11}$$

where $\Delta^{-1}_{\ell_{1\to n}}\Delta^{-1}_{q_{1\to t}} u(k)\Big|_{\frac{1}{q_1^{m_1}}\sim\frac{1}{q_t^{m_t}}}^{q_1\sim q_t\sim\big(\hat{\ell}_{i+1}(k-\sum\limits_{J=1}^{i} r_J\ell_J)+\sum\limits_{J=i+2}^{n}\ell_J\big)}$

$$= \Delta^{-1}_{\ell_{1\to n}}\Delta^{-1}_{q_{1\to t}} u(k)\Big|_{\frac{k}{q_1^{m_1}}}^{q_1k}\Big|_{\frac{k}{q_2^{m_2}}}^{q_2k}\cdots\Big|_{\frac{k}{q_t^{m_t}}}^{q_tk}\Big|^{\big(\hat{\ell}_{i+1}(k-\sum\limits_{J=1}^{i} r_J\ell_J)+\sum\limits_{J=i+2}^{n}\ell_J\big)}.$$

*Proof.* Replacing $q, m$ by $q_2, m_2$ in (9), we get

$$u(k) + u\Big(\frac{k}{q_2}\Big) + u\Big(\frac{k}{q_2^2}\Big) + \cdots + u\Big(\frac{k}{q_2^{m_2}}\Big) = \Delta_{q_2}^{-1}u(q_2k) - \Delta_{q_2}^{-1}u\Big(\frac{k}{q_2^{m_2}}\Big). \tag{12}$$

Again replacing $k$ by $k/q_1^{h_1}$ for $h_1 = 1, 2, \cdots, m_1$ in (12) , we find

$$u\Big(\frac{k}{q_1^{h_1}}\Big) + u\Big(\frac{k}{q_1^{h_1}q_2}\Big) + u\Big(\frac{k}{q_1^{h_1}q_2^2}\Big) + \cdots + u\Big(\frac{k}{q_1^{h_1}q_2^{m_2}}\Big) = \Delta_{q_2}^{-1}u\Big(\frac{q_2k}{q_1^{h_1}}\Big) - \Delta_{q_2}^{-1}u\Big(\frac{k}{q_1^{h_1}q_2^{m_2}}\Big). \tag{13}$$

Adding (12) and (13) for $h_1 = 1, 2, \cdots, m_1$ and applying (9), we arrive

$$\sum_{h_1=0}^{m_1}\sum_{h_2=0}^{m_2} u\Big(\frac{k}{q_1^{h_1}q_2^{h_2}}\Big) = \Delta_{q_1}^{-1}\Delta_{q_2}^{-1}\left\{u(q_1q_2k) - u\Big(\frac{q_2\ k}{q_1^{m_1}}\Big)\right\}$$
$$-\Delta_{q_1}^{-1}\Delta_{q_2}^{-1}\left\{u\Big(\frac{q_2\ k}{q_1^{m_1}}\Big) - u\Big(\frac{k}{q_1^{h_1}q_2^{m_2}}\Big)\right\}. \tag{14}$$

For $r_1 = 1, 2, 3, \cdots, [k/\ell_1]$, replacing $k$ by $k - r_1\ell_1$ in (14), we find that

$$\sum_{h_1=0}^{m_1}\sum_{h_2=0}^{m_2} u\Big(\frac{k-r_1\ell_1}{q_1^{h_1}q_2^{h_2}}\Big) = \Delta_{q_1}^{-1}\Delta_{q_2}^{-1}\left\{u(q_1q_2(k-r_1\ell_1)) - u\Big(\frac{q_2\ (k-r_1\ell_1)}{q_1^{m_1}}\Big)\right\}$$
$$-\Delta_{q_1}^{-1}\Delta_{q_2}^{-1}\left\{u\Big(\frac{q_2\ (k-r_1\ell_1)}{q_1^{m_1}}\Big) - u\Big(\frac{k-r_1\ell_1}{q_1^{h_1}q_2^{m_2}}\Big)\right\}. \tag{15}$$

Adding (14) and (15), we obtain

$$\sum_{r_1=0}^{[k/\ell_1]}\sum_{h_1=0}^{m_1}\sum_{h_2=0}^{m_2} u\Big(\frac{k-r_1\ell_1}{q_1^{h_1}q_2^{h_2}}\Big) = \sum_{r_1=0}^{[k/\ell_1]}\Delta_{q_1}^{-1}\Delta_{q_2}^{-1}\left\{u(q_1q_2(k-r_1\ell_1)) - u\Big(\frac{q_2\ (k-r_1\ell_1)}{q_1^{m_1}}\Big)\right\}$$
$$-\sum_{r_1=0}^{[k/\ell_1]}\Delta_{q_1}^{-1}\Delta_{q_2}^{-1}\left\{u\Big(\frac{q_2\ (k-r_1\ell_1)}{q_1^{m_1}}\Big) - u\Big(\frac{k-r_1\ell_1}{q_1^{h_1}q_2^{m_2}}\Big)\right\}. \tag{16}$$

Now, applying (5) in (16), we get

$$\sum_{r_1=0}^{[k/\ell_1]}\sum_{h_1=0}^{m_1}\sum_{h_2=0}^{m_2} u\Big(\frac{k-r_1\ell_1}{q_1^{h_1}q_2^{h_2}}\Big) = \Delta_{\ell_1}^{-1}\Delta_{q_1}^{-1}\Delta_{q_2}^{-1}u(k)\Big|_{\frac{1}{q_1^{m_1}}\sim\frac{1}{q_2^{m_2}}}^{q_1\sim q_2\sim(k+\ell_1)}. \tag{17}$$

Replacing $r_1, \ell_1$ by $r_2, \ell_2$ in (16) and repeating the above procedure, we get

$$\sum_{r_1=0}^{[\frac{k}{\ell_1}]}\sum_{r_2=0}^{[\frac{k-r_1\ell_1}{\ell_2}]}\sum_{h_1=0}^{m_1}\sum_{h_2=0}^{m_2} u\Big(\frac{k-r_1\ell_1-r_2\ell_2}{q_1^{h_1}q_2^{h_2}}\Big)+\sum_{r_1=0}^{[\frac{k}{\ell_1}]}\Delta_{\ell_2}^{-1}\Delta_{q_1}^{-1}\Delta_{q_2}^{-1}u(k)\Big|_{\frac{1}{q_1^{m_1}}\sim\frac{1}{q_2^{m_2}}}^{q_1\sim q_2\sim\hat{\ell_2}(k-r_1\ell_1)}$$
$$= \Delta_{\ell_1}^{-1}\Delta_{\ell_2}^{-1}\Delta_{q_1}^{-1}\Delta_{q_2}^{-1}u(k)\Big|_{\frac{1}{q_1^{m_1}}\sim\frac{1}{q_2^{m_2}}\sim(\hat{\ell_1}(k)+\ell_2)}^{q_1\sim q_2\sim(k+\ell_1+\ell_2)}. \tag{18}$$

Also replacing $q_1, q_2, m_1, m_2, h_1, h_2$ by $q_2, q_3, m_2, m_3, h_2, h_3$ in (14), and then $k$ by $k/q_1^{h_1}$ for $h_1 = 1, 2, \cdots, m_1$ and so adding all the resultant expressions, we arrive

$$\sum_{h_1=0}^{m_1}\sum_{h_2=0}^{m_2}\sum_{h_3=0}^{m_3} u\left(\frac{k}{q_1^{h_1}q_2^{h_2}q_3^{h_3}}\right) = \Delta_{q_1}^{-1}\Delta_{q_2}^{-1}\Delta_{q_3}^{-1}u\,(q_1q_2q_3k) - \Delta_{q_1}^{-1}\Delta_{q_2}^{-1}\Delta_{q_3}^{-1}u\left(\frac{q_2q_3k}{q_1^{m_1}}\right)$$
$$-\sum_{h_1=0}^{m_1}\Delta_{q_2}^{-1}\Delta_{q_3}^{-1}u\left(\frac{q_3k}{q_1^{h_1}q_2^{m_2}}\right) - \sum_{h_1=0}^{m_1}\sum_{h_2=0}^{m_2}\Delta_{q_3}^{-1}u\left(\frac{k}{q_1^{h_1}q_2^{h_2}q_3^{m_3}}\right). \tag{19}$$

Again using the above steps on (19) as we use to get the equations from (15) to (18), we obtain

$$\sum_{r_1=0}^{[\frac{k}{\ell_1}]}\sum_{r_2=0}^{[\frac{k-r_1\ell_1}{\ell_2}]}\sum_{h_1=0}^{m_1}\sum_{h_2=0}^{m_2}\sum_{h_3=0}^{m_3} u\Big(\frac{k-r_1\ell_1-r_2\ell_2}{q_1^{h_1}q_2^{h_2}q_3^{h_3}}\Big)+\sum_{r_1=0}^{[\frac{k}{\ell_1}]}\underset{\ell_2}{\overset{-1}{\Delta}}\underset{q_1}{\overset{-1}{\Delta}}\underset{q_2}{\overset{-1}{\Delta}}\underset{q_3}{\overset{-1}{\Delta}} u(k)\Big|_{\frac{1}{q_1^{m_1}}\sim\frac{1}{q_2^{m_2}}\sim\frac{1}{q_3^{m_3}}}^{q_1\sim q_2\sim q_3\sim\hat{\ell_2}(k-r_1\ell_1)}$$

$$=\underset{\ell_1}{\overset{-1}{\Delta}}\underset{\ell_2}{\overset{-1}{\Delta}}\underset{q_1}{\overset{-1}{\Delta}}\underset{q_2}{\overset{-1}{\Delta}}\underset{q_3}{\overset{-1}{\Delta}} u(k)\Big|_{\frac{1}{q_1^{m_1}}\sim\frac{1}{q_2^{m_2}}\sim\frac{1}{q_3^{m_3}}\sim(\hat{\ell_1}(k)+\ell_2)}^{q_1\sim q_2\sim q_3\sim(k+\ell_1+\ell_2)}. \tag{20}$$

Similarly, replacing $k$ by $k/q_1^{h_1}$ for $h_1 = 1, 2, \cdots, m_1$, $(t-3)$ times in (19) and repeating the above procedure, we get

$$\sum_{i=1}^{t-1}\sum_{(h)_{1\to i}}^{m}\underset{q_{i+1\to t}}{\overset{-1}{\Delta}} u\left(\frac{\prod\limits_{p=i+2}^{t} q_p k}{\prod\limits_{p=1}^{i} q_p^{h_p} q_{i+1}^{m}}\right)+\sum_{(h)_{1\to t}}^{m} u\left(\frac{k}{\prod\limits_{p=1}^{t} q_p^{h_p}}\right)=\underset{q_{1\to t}}{\overset{-1}{\Delta}} u\Big(\prod_{p=1}^{t} q_p k\Big)-\underset{q_{1\to t}}{\overset{-1}{\Delta}} u\left(\prod_{p=2}^{t}\frac{q_p k}{q_1^{m_1}}\right) \tag{21}$$

Now, replacing $k$ by $k-r_1\ell_1$ for $r_1 = 1, 2, 3, \cdots, [k/\ell_1]$ and applying the above procedure $'n'$ times successively, we get the desired result. □

**Remark 3.3.** *Note that, throughout this paper, to evaluate the expressions consisting of $\Delta_{\ell_i}^{-1}\Delta_{q_i}^{-1}$, first apply all the limits in terms of $'q_i'$, then operating $\Delta_{q_i}^{-1}$ and then operating $\Delta_{\ell_i}^{-1}$ and at last apply limits in terms of $'\ell_i'$.*

**Corollary 3.4.** *Let $u(k) = k^2$, $k \in [0, \infty)$. Then we obtain*

$$\sum_{r_1=0}^{[k/\ell_1]}\sum_{h_1=0}^{m_1}\sum_{h_2=0}^{m_2}\Big(\frac{k-r_1\ell_1}{q_1^{h_1}q_2^{h_2}}\Big)^2=\Delta_{\ell_1}^{-1}\Delta_{q_1}^{-1}\Delta_{q_2}^{-1}k^2\Big|_{\frac{1}{q_1^{m_1}}\sim\frac{1}{q_2^{m_2}}\sim(\hat{\ell_1}(k))}^{q_1\sim q_2\sim(k+\ell_1)}. \tag{22}$$

*Proof.* The proof is trivial by putting $t = 2$ and $n = 1$ in (11). □

The example given below illustrates Corollary 3.4.

**Example 3.5.** *Taking $m_1 = 2, m_2 = 1, \ell_1 = 6, q_1 = 3, q_2 = 4$ and $k = 10$ in (22),*

we get $\displaystyle\sum_{r_1=0}^{[k/\ell_1]}\sum_{h_1=0}^{m_1}\sum_{h_2=0}^{m_2}\Big(\frac{k-r_1\ell_1}{q_1^{h_1}q_2^{h_2}}\Big)^2=\sum_{r_1=0}^{1}\sum_{h_1=0}^{2}\sum_{h_2=0}^{1}\Big(\frac{k-r_1\ell_1}{q_1^{h_1}q_2^{h_2}}\Big)^2=138.46604942.$
By (1), we find $\Delta_{q_2}^{-1}k^2=\dfrac{k^2}{(q_2^2-1)}$ and so $\Delta_{q_1}^{-1}\Delta_{q_2}^{-1}k^2=\dfrac{k^2}{(q_1^2-1)(q_2^2-1)}$.
Also, by using the formula for $k^n$ from (3), we obtain

$$\Delta_{\ell_1}^{-1}\Delta_{q_1}^{-1}\Delta_{q_2}^{-1}k^2=\Delta_{\ell_1}^{-1}\frac{\ell_1 k_{\ell_1}^{(1)}+k_{\ell_1}^{(2)}}{(q_1^2-1)(q_2^2-1)}. \tag{23}$$

Using the notation given in the Preliminaries section and by Remark 3.4, we write

$$\Delta_{\ell_1}^{-1}\Delta_{q_1}^{-1}\Delta_{q_2}^{-1}k^2\Big|_{\frac{1}{q_1^{m_1}}\sim\frac{1}{q_2^{m_2}}\sim(\hat{\ell}_1(k))}^{q_1\sim q_2\sim(k+\ell_1)} = \Delta_{\ell_1}^{-1}\Delta_{q_1}^{-1}\Delta_{q_2}^{-1}k^2\Big|_{\frac{k}{q_1^{m_1}}}^{q_1 k}\Big|_{\frac{k}{q_2^{m_2}}}^{q_2 k}\Big|_{\hat{\ell}_1(k)}^{k+\ell_1}$$
$$= \Delta_{\ell_1}^{-1}\Delta_{q_1}^{-1}\Delta_{q_2}^{-1}(q_1^2-\frac{1}{q_1^{2m_1}})(q_2^2-\frac{1}{q_2^{2m_2}})k^2\Big|_{\hat{\ell}_1(k)}^{k+\ell_1}.$$

Using (4) in (23) and by applying the limits for k, the above equation becomes

$$\Delta_{\ell_1}^{-1}\Delta_{q_1}^{-1}\Delta_{q_2}^{-1}k^2\Big|_{\frac{1}{q_1^{m_1}}\sim\frac{1}{q_2^{m_2}}\sim(\hat{\ell}_1(k))}^{q_1\sim q_2\sim(k+\ell_1)} = \frac{(q_1^2-\frac{1}{q_1^{2m_1}})(q_2^2-\frac{1}{q_2^{2m_2}})}{(q_1^2-1)(q_2^2-1)}\Big(\frac{k_{\ell_1}^{(2)}}{2}+\frac{k_{\ell_1}^{(3)}}{3\ell_1}\Big)\Big|_{\hat{\ell}_1(k)}^{k+\ell_1}$$
$$= 138.46604942.$$

**Theorem 3.6.** *[q_ℓ mixed higher order Summation Formula] Let $k\in[0,\infty)$, $q\neq 0$ and $\ell>0$. Then the equation $\Delta_\ell^n\Delta_q^t v(k)=u(k)$ has a solution of the form*

$$\sum_{r=0}^{[k/\ell]}\sum_{h=0}^{m}\binom{h+t-1}{t-1}\binom{r+n-1}{n-1}u\Big(\frac{k-r\ell}{q^h}\Big)$$
$$+\sum_{r=1}^{n-1}\frac{([\frac{k}{\ell}]+r)^{(r)}}{r!}\Delta_\ell^{-(n-r)}\Delta_q^{-t}u(q^{t-1}k)\Big|_{\frac{1}{q^m}}^{q\sim\big(\hat{\ell}(k)+(n-r-1)\ell\big)}$$
$$+\sum_{h=1}^{t-1}\frac{(m+h)^{(h)}}{h!}\Delta_\ell^{-n}\Delta_q^{-(t-h)}u\Big(\frac{q^{t-(h+1)}k}{q^m}\Big)\Big|_{\hat{\ell}(k)+(n-1)\ell}^{k+n\ell}$$
$$-\sum_{h=1}^{t-1}\sum_{r=1}^{n-1}\frac{(m+h)^{(h)}}{h!}\frac{([\frac{k}{\ell}]+r)^{(r)}}{r!}\Delta_\ell^{-(n-r)}\Delta_q^{-(t-h)}u\Big(\frac{q^{t-(h+1)}k}{q^m}\Big)\Big|^{\hat{\ell}(k)+(n-r-1)\ell}$$
$$=\Delta_\ell^{-n}\Delta_q^{-t}u(q^{t-1}k)\Big|_{\frac{1}{q^m}\sim\big(\hat{\ell}(k)+(n-1)\ell\big)}^{q\sim(k+n\ell)}. \tag{24}$$

*Proof.* From (9), we have

$$u(k)+u\Big(\frac{k}{q}\Big)+u\Big(\frac{k}{q^2}\Big)+\cdots+u\Big(\frac{k}{q^m}\Big)=\Delta_q^{-1}u(qk)-\Delta_q^{-1}u\Big(\frac{k}{q^m}\Big). \tag{25}$$

Replacing $k$ by $k/q^h$ for $h=1,2,\cdots,m$ in (25) and then summing all the resultant expressions with (25), we arrive

$$\sum_{h=0}^{m}\binom{h+1}{1}u\Big(\frac{k}{q^h}\Big)=\Delta_q^{-2}u\big(q^2k\big)-\Delta_q^{-2}u\Big(\frac{qk}{q^m}\Big)-(m+1)\Delta_q^{-1}u\Big(\frac{k}{q^m}\Big). \tag{26}$$

Also replacing $k$ by $k-r\ell$ for $r=1,2,\cdots,[k/\ell]$ in (26) and then adding all the resultant expressions with (26), we arrive

$$\sum_{h=0}^{m}\binom{h+1}{1}\sum_{r=0}^{[k/\ell]}u\Big(\frac{k-r\ell}{q^h}\Big)=\sum_{r=0}^{[k/\ell]}\Big\{\Delta_q^{-2}u\big(q^2(k-r\ell)\big)-\Delta_q^{-2}u\Big(\frac{q(k-r\ell)}{q^m}\Big)$$

$$-(m+1)\Delta_q^{-1}u\Big(\frac{k-r\ell}{q^m}\Big)\Big\}. \quad (27)$$

By (5), (27) becomes

$$\sum_{h=0}^{m}\binom{h+1}{1}\sum_{r=0}^{[k/\ell]}u\Big(\frac{k-r\ell}{q^h}\Big)=\Delta_\ell^{-1}\Delta_q^{-2}\Big\{u\big(q^2(k+\ell)\big)-u\big(q^2\hat{\ell}(k)\big)\Big\}-\Delta_\ell^{-1}\Delta_q^{-2}$$

$$\Big\{u\Big(\frac{q(k+\ell)}{q^m}\Big)-u\Big(\frac{q\hat{\ell}(k)}{q^m}\Big)\Big\}-(m+1)\Delta_\ell^{-1}\Delta_q^{-1}\Big\{u\Big(\frac{k+\ell}{q^m}\Big)-u\Big(\frac{\hat{\ell}(k)}{q^m}\Big)\Big\}, \quad (28)$$

which yields

$$\sum_{h=0}^{m}\binom{h+1}{1}\sum_{r=0}^{[k/\ell]}u\Big(\frac{k-r\ell}{q^h}\Big)=\Delta_\ell^{-1}\Delta_q^{-2}u(qk)\Big|_{\frac{1}{q^m}\sim\hat{\ell}(k)}^{q\sim(k+\ell)}-(m+1)\Delta_\ell^{-1}\Delta_q^{-1}u\Big(\frac{k}{q^m}\Big)\Big|_{\hat{\ell}(k)}^{k+\ell}.$$

Again replacing $k$ by $k-r\ell$ for $r=1,2,\cdots,[k/\ell]$ in (28) and summing all the resultant expressions, we obtain

$$\sum_{h=0}^{m}\binom{h+1}{1}\sum_{r=0}^{[k/\ell]}\binom{r+1}{1}u\Big(\frac{k-r\ell}{q^h}\Big)=\Delta_\ell^{-2}\Delta_q^{-2}u(qk)\Big|_{\frac{1}{q^m}\sim(\hat{\ell}(k)+\ell)}^{q\sim(k+2\ell)}$$

$$-(m+1)\Delta_\ell^{-2}\Delta_q^{-1}u\Big(\frac{k}{q^m}\Big)\Big|_{\hat{\ell}(k)+\ell}^{k+2\ell}-\Big([\frac{k}{\ell}]+1\Big)\Delta_\ell^{-1}\Delta_q^{-2}u(qk)\Big|_{\frac{1}{q^m}}^{q\sim\hat{\ell}(k)}$$

$$+(m+1)\Big([\frac{k}{\ell}]+1\Big)\Delta_\ell^{-1}\Delta_q^{-1}u\Big(\frac{k}{q^m}\Big)\Big|^{\hat{\ell}(k)}. \quad (29)$$

Replacing $k$ by $k/q^h$ for $h=1,2,\cdots,m$ in (26) and then summing all the resultant expressions with (26), we arrive

$$\sum_{h=0}^{m}\binom{h+2}{2}u\Big(\frac{k}{q^h}\Big)=\Delta_q^{-3}u(q^3k)-\Delta_q^{-3}u\Big(\frac{q^2k}{q^m}\Big)-(m+1)\Delta_q^{-2}u\Big(\frac{qk}{q^m}\Big)$$

$$-\frac{(m+2)^{(2)}}{2}\Delta_q^{-1}u\Big(\frac{k}{q^m}\Big). \quad (30)$$

Continuing in this way, we obtain

$$\sum_{h=0}^{m}\binom{h+t-1}{t-1}u\Big(\frac{k}{q^h}\Big)+\sum_{h=1}^{t-1}\frac{(m+h)^{(h)}}{h!}\Delta_q^{-(t-h)}u\Big(\frac{q^{t-(h+1)}k}{q^m}\Big)$$

$$=\Delta_q^{-t}u(q^tk)-\Delta_q^{-t}u\Big(\frac{q^{t-1}k}{q^m}\Big). \quad (31)$$

And then replacing $k$ by $k-r\ell$ for $r=1,2,\cdots,[k/\ell]$ in (31) and using the above procedure with $\Delta_\ell^{-1}$, 'n' times repeatedly, we get the proof of the theorem. □

**Corollary 3.7.** *For any bounded real valued function $u(k)$ and $q \neq 0$, we obtain*

$$\sum_{r=0}^{[k/\ell]}\sum_{h=0}^{m}\binom{h+3}{3}\binom{r+1}{1}u\Big(\frac{k-r\ell}{q^h}\Big)+\big(\big[\tfrac{k}{\ell}\big]+1\big)\Delta_\ell^{-1}\Delta_q^{-4}u(q^3k)\Big|_{\frac{1}{q^m}}^{q\sim\hat{\ell}(k)}$$

$$+\sum_{h=1}^{3}\frac{(m+h)^{(h)}}{h!}\Delta_\ell^{-2}\Delta_q^{-(4-h)}u\Big(\frac{q^{4-(h+1)}k}{q^m}\Big)\Big|_{(\hat{\ell}(k)+\ell)}^{k+2\ell}-\sum_{h=1}^{3}\frac{(m+h)^{(h)}}{h!}\big(\big[\tfrac{k}{\ell}\big]+1\big)$$

$$\Delta_\ell^{-1}\Delta_q^{-(4-h)}u\Big(\frac{q^{4-(h+1)}k}{q^m}\Big)\Big|^{\hat{\ell}(k)}=\Delta_\ell^{-2}\Delta_q^{-4}u(q^3k)\Big|_{\frac{1}{q^m}\sim(\hat{\ell}(k)+\ell)}^{q\sim(k+2\ell)}. \tag{32}$$

*Proof.* The proof is obvious by putting $t=4$ and $n=2$ in (24). □

The following example is a verification of Corollary 3.7.

**Example 3.8.** *Consider $u(k)=k_q^{(2)}, m=2, k=15, \ell=8$ and $q=5$ in (32). Then using the procedure in Example 3.5, we get*

$$\Delta_\ell^{-1}\Delta_q^{-4}u(q^3k)\Big|_{\frac{1}{q^m}}^{q\sim\hat{\ell}(k)}=\Delta_\ell^{-1}\Delta_q^{-4}\left\{(q^8-q^2)k^2-(q^5-q^2)k)\right\}\Big|^{\hat{\ell}(k)}$$

$$=\left\{\frac{(q^8-q^2)}{(q^2-1)^4}\Big(\frac{k_\ell^{(2)}}{2}+\frac{k_\ell^{(3)}}{3\ell}\Big)-\frac{(q^5-q^2)k_\ell^{(2)}}{2\ell(q-1)^4}\right\}\Bigg|^{7},$$

$$\sum_{h=1}^{3}\frac{(m+h)^{(h)}}{h!}\Delta_\ell^{-2}\Delta_q^{-(4-h)}u\Big(\frac{q^{4-(h+1)}k}{q^m}\Big)\Big|_{\hat{\ell}(k)+\ell}^{k+2\ell}=\Bigg\{\Big(\frac{3}{(q^2-1)^3}+\frac{6}{q^2(q^2-1)^2}$$

$$+\frac{10}{q^4(q^2-1)}\Big)\Big(\frac{k_\ell^{(3)}}{6\ell}+\frac{k_\ell^{(4)}}{12\ell^2}\Big)-\Big(\frac{3q}{(q-1)^3}+\frac{6}{(q-1)^2}+\frac{10}{q(q-1)}\Big)\frac{k_\ell^{(3)}}{6\ell^2}\Bigg\}\Bigg|_{15}^{31},$$

$$\sum_{h=1}^{3}\frac{(m+h)^{(h)}}{h!}\Delta_\ell^{-1}\Delta_q^{-(4-h)}u\Big(\frac{q^{4-(h+1)}k}{q^m}\Big)\Big|^{\hat{\ell}(k)}=\Bigg\{\Big(\frac{3}{(q^2-1)^3}+\frac{6}{q^2(q^2-1)^2}$$

$$+\frac{10}{q^4(q^2-1)}\Big)\Big(\frac{k_\ell^{(2)}}{2}+\frac{k_\ell^{(3)}}{3\ell}\Big)-\Big(\frac{3q}{(q-1)^3}+\frac{6}{(q-1)^2}+\frac{10}{q(q-1)}\Big)\frac{k_\ell^{(2)}}{2\ell}\Bigg\}\Bigg|^{7}$$

and

$$\Delta_\ell^{-2}\Delta_q^{-4}u(q^3k)\Big|_{\frac{1}{q^m}\sim(\hat{\ell}(k)+\ell)}^{q\sim(k+2\ell)}=\left\{\frac{(q^8-q^2)}{(q^2-1)^4}\Big(\frac{k_\ell^{(3)}}{6\ell}+\frac{k_\ell^{(4)}}{12\ell^2}\Big)-\frac{(q^5-q^2)k_\ell^{(3)}}{6\ell^2(q-1)^4}\right\}\Bigg|_{15}^{31}.$$

Putting these values in (32) and by applying the limits for 'k', we get

$$\sum_{r=0}^{1}\sum_{h=0}^{2}\binom{h+3}{3}\binom{r+1}{1}\Big(\frac{k-8r}{5^h}\Big)_5^{(2)}+2\Delta_\ell^{-1}\Delta_q^{-4}(q^3k)_5^{(2)}\Big|_{\frac{1}{q^m}}^{q\sim\hat{\ell}(k)}$$

$$+\sum_{h=1}^{3}\frac{(2+h)^{(h)}}{h!}\Delta_{\ell}^{-2}\Delta_{q}^{-(4-h)}\Big(\frac{q^{4-(h+1)}k}{q^{m}}\Big)_{5}^{(2)}\Big|_{\hat{\ell}(k)+\ell}^{k+2\ell}-2\sum_{h=1}^{3}\frac{(2+h)^{(h)}}{h!}$$

$$\Delta_{\ell}^{-1}\Delta_{q}^{-(4-h)}\Big(\frac{q^{4-(h+1)}k}{q^{2}}\Big)_{5}^{(2)}\Big|^{\hat{\ell}(k)}=\Delta_{\ell}^{-2}\Delta_{q}^{-4}(q^{3}k)_{5}^{(2)}\Big|_{\frac{1}{q^{m}}\sim\left(\hat{\ell}(k)+\ell\right)}^{q\sim(k+2\ell)}=37.6315646.$$

## 4. Conclusion

In this paper, we have derived multi-series summation formula for mixed $q$-$\ell$ difference equation. Moreover, solution for higher order mixed difference equation is also obtained. Consequently, relevant examples are being given to verify the results.

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