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Some New Results on Total Equitable Domination in Graphs

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Abstract. A dominating set D is total if every vertex of V is adjacent to at least one vertex of D while D is called equitable if for every v in $V - D$ there exists a vertex u in D such that uv in E and $|d(u) - d(v)| \leq 1$. A dominating set which is both total and equitable is called total equitable dominating set The total equitable domination number of G is the minimum cardinality of a total equitable dominating set of G which is denoted by $\gamma_t^e(G)$. In this paper we contribute some general results on this concept.

Key words: Dominating set, equitable dominating set, total dominating set.

AMS Subject classification: 05C69

1. Introduction

We begin with a finite, connected and undirected graph $G = (V, E)$ without loops and multiple edges. For a vertex $v \in V$, the open neighborhood $N(v)$ of v is defined as $N(v) = \{u \in V : uv \in E\}$. For any undefined term and notation in graph theory we rely upon West [9] while for any undefined term related to theory of domination we refer to Haynes et al. [4]

The study of dominating sets in graph and its related concepts is getting momentum due to its diversified applications for the solution of many real life problems. An excellent discussion on theory of dominating sets is carried out by

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Haynes *et al.* [4] while the edited work by the same authors on some advanced topics of domination in graphs can be found in [5]. For a graph of order n, the set $D \subseteq V$ of vertices is called a dominating set if every vertex $v \in V$ is either an element of D or is adjacent to an element of D. The minimum cardinality of a dominating set of G is called the domination number of G which is denoted by $\gamma(G)$ and the corresponding dominating set is called a γ-set of G. There are many variants of dominating sets such as total dominating set, global dominating set, equitable dominating set, fractional dominating set, etc. are among mention a few. A subset D of V is called a total dominating set of G if $N(D) = V$ or if every vertex $v \in V$ is adjacent to at least one element in D. The minimum cardinality of total dominating set is called total domination number which is denoted by $\gamma_t(G)$. This concept was introduced by Cockayne *et al.* [3]. A variant of total dominating set is also introduced in recent past by Sivagnanam [6]. A subset D of V is called an equitable dominating set if for every $v \in V - D$ there exists a vertex $u \in D$ such that $uv \in E$ and $|d(v) - d(u)| \leqslant 1$, where $d(u)$ denotes the degree of vertex u and $d(v)$ denotes the degree of vertex v. The minimum cardinality of such a dominating set is called the equitable domination number of G which is denoted by $\gamma^e(G)$. This concept was introduced by Swaminathan and Dharmalingam [7]. Some variants of equitable domination like global equitable domination [2], equi independent equitable domination [8], total equitable domination [1] are also available in the literature. The present work is intended to report some investigations on the concept of total equitable domination in graph. A dominating set which is both total and equitable is called total equitable dominating set. The total equitable domination number of G is the minimum cardinality of a total equitable dominating set of G which is denoted by $\gamma_t^e(G)$. This concept was introduced by Basavanagoud *et al.* [1]. In the same paper they have proved some fundamental results related to this newly defined concepts.

In the present work we determine the bounds of $\gamma_t^e(G)$ in terms of various graph theoretic parameters like open packing, open boundary and private numbers(internal and external). As proved by Basavanagoud *et al.* [1].

The open boundary of $D \subseteq V$ is denoted by $\mathfrak{B}(D)$ and defined as $\mathfrak{B}(D) = \{v : |N(v) \cap D| = 1\}$ while the private number of $v \in D$ is denoted by $pn(v, D)$ and defined as $pn(v, D) = \{w \in V : N(w) \cap D = \{v\}\}$.

Proposition 1.1. [1] *For any graph G without isolated vertices, an equitable total dominating set D is minimal if and only if for every $u \in D$, one of the following two properties holds:*

(1) *There exists a vertex $v \in V - D$ such that $N(v) \cap D = \{u\}$, $|d(u) - d(v)| \leqslant 1$*
(2) *$\langle D - \{u\}\rangle$ contains no isolated vertices.*

Proposition 1.2. [1] *For any connected graph G(except $K_{m,n}$ with $|m-n| \geqslant 2$ and $m, n \geqslant 2$) contains no isolated vertices and $\Delta(G) < n - 1$, then $\gamma_t^e(G) \leqslant n - \Delta(G)$.*

Proposition 1.3. [1] *For any complete bipartite graph $K_{m,n}$,*

$$\gamma_t^e(K_{m,n}) = \begin{cases} 2 & ;\ \text{if } |m-n| \leq 1,\ 1 \leq m \leq n \\ m+n & ;\ \text{if } |m-n| \geq 2,\ m, n \geq 2 \end{cases}$$

2. Main Results

Definition 2.1. *A subset $S \subseteq V$ is open packing in G, if every distinct vertices $u, v \in S$, $N(v) \cap N(u) = \phi$. The open packing number $\rho^0(G)$ is the maximum cardinality of an open packing in G.*

Theorem 2.2. *If D is an open packing set of a graph $G \neq K_{m,n}$ then $V - D$ is a total equitable dominating set.*

Proof. Let D be any open packing set of G. Then all the components of induced subgraph $\langle D\rangle$ are either K_1 or K_2. This implies that for every vertex $v \in D$, there exists $u \in V - D$ such that $uv \in E$ as $\delta(G) \geqslant 2$.

Further, $u \in V - D$ is adjacent to the vertices of either of the components of $\langle D\rangle$, it follows that $|d(u) - d(v)| \leqslant 1$

Moreover the induced subgraph $\langle V - D \rangle$ has no isolated vertices as $\delta(G) \geqslant 2$. It means that $V - D$ is a total equitable dominating set in G. □

Corollary 2.3. *For a graph $G \neq K_{m,n}$ of order n with $\delta(G) \geqslant 2$ and $\rho^0(G)$ be an open packing of G then $\gamma_t^e(G) \leqslant n - \rho^0(G)$.*

Proof. Let D be any open packing set of G. Then $V - D$ is a total equitable dominating set of G. It is obvious that for any $D \subseteq V$, $|D| \leqslant |V| - |V - D|$. This implies $|V - D| \leqslant |V| - |D|$. Therefore $\gamma_t^e(G) \leqslant n - \rho^0(G)$. □

Remark 2.4. *For complete bipartite graph $K_{m,n}$, $\gamma_t^e(K_{m,n}) \nleq |V(K_{m,n})| - \rho^0(K_{m,n})$ as $\gamma_t^e(K_{m,n}) = |V(K_{m,n})| = m + n$. Consequently $\rho^0(K_{m,n}) \leq 0$.*

Theorem 2.5. *Let D be total equitable dominating set in graph G. Then D is a minimal total equitable dominating set in G if and only if $pn(v, D) \neq \phi$, for all $v \in D$.*

Proof. Let D be a minimal total equitable dominating set in G and $v \in D$. If possible $pn(v, D) = \phi$. Then there exists a vertex $u \in V$ such that $N(u) \cap D \neq \{v\}$. Consequently $u \in V$ is not dominates of any vertex in D. This contradicts the minimality of D. Therefore $pn(v, D) \neq \phi$, for all $v \in D$.

Conversely, let $pn(v, D) \neq \phi$, for all $v \in D$. Thus $D - \{v\}$ is not total equitable dominating set in G, it follows that D is minimal total equitable dominating set in G. □

Theorem 2.6. *A total equitable dominating set D in a graph G is a minimal total equitable dominating set if and only if $\mathfrak{B}(D)$ totally dominates D.*

Proof. First we assume that D is minimal total equitable dominating set in G. By Theorem 2.3, $pn(v, D) \neq \phi$, for all $v \in D$. If $pn(v, D) - D \neq \phi$ then there exists every vertex $u \in V - D$ such that $N(u) \cap D = \{v\}$. It implies that $|N(u) \cap D| = 1$ then $u \in \mathfrak{B}(D)$. If $pn(v, D) \cap D \neq \phi$ then every vertex $u \in D$, $u \in \mathfrak{B}(D)$ as $N(u) \cap D = \{v\}$. Hence $\mathfrak{B}(D)$ totally dominates D.

Conversely, we assume that $\mathfrak{B}(D)$ totally dominates D. That is, $\mathfrak{B}(D) \subseteq D$. Let $v \in D$ and $u \in \mathfrak{B}(D)$ be such that $uv \in E$. We claim that D is a minimal total equitable dominating set in graph G. If $u \in D$ then $u \in pn(v,D) \cap D$ and if $u \notin D$ then $u \in pn(v,D) - D$. Thus for all the vertices $v \in D$, $pn(v,D) \neq \phi$. Hence by Theorem 2.3, D is a minimal total equitable dominating set in G. $\square$

Definition 2.7. *For a subset D of V with $|D| = M$, the generalized maximum degree of graph G is denoted by $\Delta_M(G)$ and defined as $\Delta_M(G) = \max\{|N(D)|\}$. i.e. $|N(D)| \leqslant \Delta_M(G)$.*

Basavanagoud *et al.* [1] have proved that $\gamma_t^e(G) \leqslant n - \Delta(G)$ while the following theorem gives a necessary condition for which the bound is sharp.

Theorem 2.8. *Let G be a graph of order $n \geqslant 3$ with $\Delta(G) \leqslant n-2$. If $\Delta_{n-\Delta(G)-1}(G) = n-1$, then $\gamma_t^e(G) = n - \Delta(G)$.*

Proof. Let D be minimal total equitable dominating set in G. If $\Delta_{n-\Delta(G)-1}(G)$ $= n-1$. Then $|N(D)| \leqslant \Delta_{n-\Delta(G)-1}(G) < n$ for all $D \subseteq V$ and $|D| = n - \Delta(G) - 1$. That is, $\gamma_t^e(G) \geqslant n - \Delta(G)$. Also by Proposition 1.2, $\gamma_t^e(G) \leqslant n - \Delta(G)$. Hence $\gamma_t^e(G) = n - \Delta(G)$. $\square$

Observation 2.9. *If $\Delta(G) \geqslant 3$ and G is connected graph order $n \geqslant 3$ and size m, then $m \leqslant \Delta(G)(n - \gamma_t^e(G))$.*

Theorem 2.10. *Let G be a graph of odd order n and size m with $n-1$ vertices of degree three and a vertex of degree two, then $\gamma_t^e(G) \leqslant \dfrac{n-1}{2}$.*

Proof. Here, $|V| = n$ is an odd. Let $|V| = n = 2k+1$, for some $k \in \mathbb{N}$. By the fundamental theorem of graph theory $2m = \sum_{v \in V} d(v) = \sum_{v \in V} |N(v)| = 3(2k+1) - 1$. Now by Observation 2.8, $m \leqslant \Delta(D)(n - \gamma_t^e(G))$ which is equivalent to $\dfrac{(3(2k+1)-1)}{2} \leqslant 3\left((2k+1) - \gamma_t^e(G)\right)$. Consequently $\gamma_t^e(G) \leqslant k$. Hence $\gamma_t^e(G) \leqslant \dfrac{n-1}{2}$. $\square$

3. Concluding Remarks

The concept of total domination is interesting as it confirms the relation between vertices within the dominating set while the concept of equitable domination is important as it depends upon the degree of dominating vertices. The present work is based on the concept of total equitable dominating sets in graph. This domination model possesses the blends of total domination as well as equitable domination in graphs. We have derived some general results on the concept of total equitable domination in graph.

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