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Stability of Orthogonally Additive-Quadratic Functional Equation in Multi-Banach Spaces

[1]R. Murali and [2]A. Antony Raj

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Abstract. In this paper, we establish the Hyers-Ulam stability of the following Orthogonally Additive-Quadratic functional equation in Multi-Banach Spaces.

$$\zeta(2i+j) - \zeta(i+2j) - \zeta(i+j) - \zeta(j-i) - \zeta(i) + \zeta(j) + \zeta(2j) = 0$$

with $i \perp j$ where, $\perp$ is orthogonality in the sense of Ratz.

1. Introduction

The stability problem of functional equations has a long history. Stability is investigated when one concerns whether a small error of parameters causes a large deviation of the solution. Generally speaking, given a function which satisfies a functional equation approximately called a approximate solution, we ask: Is there a solution of this equation which is close to the approximate solution in some accuracy? An earlier work was done by Hyers [6] in order to answer Ulam's equation [18] on approximately additive mappings.

During last decades various stability problems for large variety of functional equations have been investigated by several mathematicians. A large list of references concerning in the stability of functional equations can be found.
e.g.([1], [2], [6], [7], [8], [10]).

In 2010, Liguang Wang, Bo Liu and ran Bai [9] proved the stability of a mixed type functional equations on Multi - Banach Spaces. In 2010, Tian Zhou Xu, John Michael Rassias, Wan Xin Xu [17] investigated the generalized Ulam-Hyers stability of the general mixed additive-quadratic-cubic-quartic functional equation

$$\begin{aligned} f(x+ny) + f(x-ny) &= n^2 f(x+y) + n^2 f(x-y) + 2(1-n^2) f(x) \\ &\quad + \frac{n^4 - n^2}{12}\left[f(2y) + f(-2y) - 4f(y) - 4f(-y)\right] \end{aligned}$$

[1]Corresponding Author: E-mail: shcrmurali@yahoo.co.in

[1,2]Department of Mathematics, Sacred Heart College, Tirupattur, TamilNadu, India - 635 601

for fixed integers n with $n \neq 0, \pm 1$ in Multi- Banach Spaces.

In 2011, Zhihua Wang, Xiaopei Li and Th. M. Rassias[21] proved the Hyers - Ulam stability of the additive - cubic - quartic functional equations

$$\begin{aligned} 11\left[f(x+2y)+f(x-2y)\right] &= 44\left[f(x+y)+f(x-y)\right]+12f(3y) \\ &\quad -48f(2y)+60f(y)-66f(x) \end{aligned}$$

in Multi - Banach Spaces by using fixed point method.

In 2013, Fridoun Moradlou [5] proved the generalized Hyers-Ulam-Rassias stability of the Euler-Lagrange-Jensen Type Additive mapping in Multi-Banach Spaces. In 2015, Xiuzhong Yang, Lidan Chang, Guofen Liu[19] estabilished the orthogonal stability of mixed additive-quadratic jensen type functional equation in Multi-Banach Spaces.

In 2015,Young Ju Jeon and Chang Il Kim [20] investigated the following additive -quadratic functional equation

$$f(2x+y)-f(x+2y)-f(x+y)-f(y-x)-f(x)+f(y)+f(2y)=0$$

in orthogonality space by using fixed point method.

In 2016, R. Murali, M. Deboral and A. Antony Raj [12] proved the Hyers-Ulam stability of the additive-cubic functional equation

$$f(2x+y)+f(2x-y)-f(4x)=2f(x+y)+2f(x-y)-8f(2x)+10f(x)-2f(-x)$$

for all x, y with $x \perp y$. in orthogonal space.

In 2016, Sattar Alizadeh, Fridoun Moradlou [16] proved the generalized Hyers-Ulam-Rassias stability of the quadratic mapping in multi-Banach spaces.

In this paper, we achieve the stability of the orthogonally Additive-Quadratic functional equation

$$\zeta(2i+j)-\zeta(i+2j)-\zeta(i+j)-\zeta(j-i)-\zeta(i)+\zeta(j)+\zeta(2j)=0 \tag{1}$$

with $i \perp j$ in Multi-Banach Spaces.

It is easy to see that the function $\zeta(i)=ai^2+bi$ is a solution of (1).

Theorem 1.1. [3], [14] *Let $(\mathcal{X}, d)$ be a complete generalized metric space and let $\mathcal{J} : \mathcal{X} \to \mathcal{X}$ be a strictly contractive mapping with Lipschitz constant $\mathcal{L} < 1$. Then for each given element $x \in \mathcal{X}$, either*

$$d(\mathcal{J}^n x, \mathcal{J}^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(i) $d(\mathcal{J}^n x, \mathcal{J}^{n+1} x) < \infty$ *for all* $n \geq n_0$;

(ii) *The sequence $\{\mathcal{J}^n x\}$ is convergent to a fixed point y^* of $\mathcal{J}$;*

(iii) *y^* is the unique fixed point of T in the set $Y = \{y \in X : d(\mathcal{J}^{n_0} x, y) < \infty\}$;*

(iv) $d(y, y^*) \leq \frac{1}{1-\mathcal{L}} d(y, \mathcal{J} y)$ *for all* $y \in Y$..

Now, let us recall some concepts concerning Multi-Banach spaces.
Let $(\wp, \|.\|)$ be a complex normed space, and let $k \in \mathbb{N}$. We denote by $\wp^k$ the linear space $\wp \oplus \wp \oplus \wp \oplus ... \oplus \wp$ consisting of $k-$ tuples $(x_1, ..., x_k)$ where $x_1, ..., x_k \in \wp$. The linear operations on $\wp^k$ are defined coordinate wise. The zero element of either $\wp$ or $\wp^k$ is denoted by 0. We denote by $\mathbb{N}_k$ the set $\{1, 2, ..., k\}$ and by Ψ_k the group of permutations on k symbols.

Definition 1.2. [4] A Multi- norm on $\left\{\wp^k : k \in \mathbb{N}\right\}$ is a sequence $(\|.\|) = (\|.\|_k : k \in \mathbb{N})$ such that $\|.\|_k$ is a norm on $\wp^k$ for each $k \in \mathbb{N}, \|x\|_1 = \|x\|$ for each $x \in \wp$, and the following axioms are satisfied for each $k \in \mathbb{N}$ with $k \geq 2$:

(1) $\left\|\left(x_{\sigma(1)}, ..., x_{\sigma(k)}\right)\right\|_k = \|(x_1 ... x_k)\|_k$, for $\sigma \in \Psi_k, x_1, ..., x_k \in \wp$;

(2) $\|(\alpha_1 x_1, ..., \alpha_k x_k)\|_k \leq (\max_{i \in \mathbb{N}_k} |\alpha_i|) \|(x_1 ... x_k)\|_k$
for $\alpha_1 ... \alpha_k \in \mathbb{C}, x_1, ..., x_k \in \wp$;

(3) $\|(x_1, ..., x_{k-1}, 0)\|_k = \|(x_1, ..., x_{k-1})\|_{k-1}$, for $x_1, ..., x_{k-1} \in \wp$;

(4) $\|(x_1, ..., x_{k-1}, x_{k-1})\|_k = \|(x_1, ..., x_{k-1})\|_{k-1}$ for $x_1, ..., x_{k-1} \in \wp$.

In this case, we say that $\left((\wp^k, \|.\|_k) : k \in \mathbb{N}\right)$ is a multi - normed space.

Suppose that $\left((\wp^k, \|.\|_k) : k \in \mathbb{N}\right)$ is a multi - normed spaces, and take $k \in \mathbb{N}$. We need the following two properties of multi - norms. They can be found in [4].

(a) $\|(x, ... x)\|_k = \|x\|$, for $x \in \wp$,

(b) $\max\limits_{i \in \mathbb{N}_k} \|x_i\| \leq \|(x_1, ..., x_k)\|_k \leq \sum\limits_{i=1}^{k} \|x_i\| \leq k \max\limits_{i \in \mathbb{N}_k} \|x_i\|$, for $x_1, ..., x_k \in \wp$.

It follows from (b) that if $(\wp, \|.\|)$ is a Banach space, then $(\wp^k, \|.\|_k)$ is a Banach space for each $k \in \mathbb{N}$; In this case, $\left((\wp^k, \|.\|_k) : k \in \mathbb{N}\right)$ is a multi - Banach space.

Lemma 1.3. [4] *Suppose that* $k \in \mathbb{N}$ *and* $(x_1 ... x_k) \in \wp^k$*. For each* $j \in \{1...k\}$, *let* $(x_n^j)_{n=1,2...}$ *be a sequence in* $\wp$ *such that* $\lim_{n\to\infty} x_n^j = x_j$*. Then*

$$\lim_{n\to\infty} \left(x_n^1 - y_1, ..., x_n^k - y_k\right) = (x_1 - y_1 ... x_k - y_k) \tag{2}$$

holds for all $(y_1, ..., y_k) \in \wp^k$.

Definition 1.4. [4] Let $\left((\wp^k, \|.\|_k) : k \in \mathbb{N}\right)$ be a multi - normed space. A sequence (x_n) in $\wp$ is a multi-null sequence if for each $\eta > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\sup_{k\in\mathbb{N}} \|(x_n, ..., x_{n+k-1})\|_k \leq \eta \qquad (n \geq n_0). \tag{3}$$

Let $x \in \wp$, we say that the sequence (x_n) is multi-convergent to x in $\wp$ and write $\lim_{n\to\infty} x_n = x$ if $(x_n - x)$ is a multi - null sequence.

There are several orthogonality notations on a real normed spaces available. But here, we present the orthogonal concept introduced by Ratz [13].
This is given in the following definition.

Definition 1.5. Suppose that X is a vector space (algebraic module) with $\dim X \geq 2$, and $\perp$ is a binary relation on X with the following properties:

(1) Totality of $\perp$ for zero: $x \perp 0, 0 \perp x$ for all $x \in X$;
(2) Independence: If $x, y \in X - \{0\}$ and $x \perp y$, then x and y are linearly independent;
(3) Homogeneity: If $x, y \in X$ and $x \perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in \mathbb{R}$;
(4) Thalesian properity: If P is a 2-dimensional subspace of X, $x \in P$ and $\lambda \in \mathbb{R}_+$ which is the set of non-negative real numbers, then there exists $y_0 \in P$ such that $x \perp y_0$ and $x + y_0 \perp \lambda x - y_0$.

The pair $(X, \perp)$ is called an orthogonality space (resp., module). By an orthogonality normed space (normed module) we mean an orthogonality space (resp., module) having a normed (resp., normed module) structure.

Definition 1.6. Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if and only if d satisfies

- $d(x, y) = 0$ if and only if $x = y$;
- $d(x, y) = d(y, x)$ for all $x, y \in X$;
- $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 1.7. *Let S be an orthogonality space and let $\big((T^k, \|.\|) : K \in \mathbb{N}\big)$ be a multi-Banach space. Suppose that η is a nonnegative real number and $\zeta : S \to T$ is a mapping satisfying*

$$\sup_{k\in\mathbb{N}} \|(D\zeta(i_1, j_1), ..., D\zeta(i_k, j_k))\|_k \leq \eta \tag{4}$$

$i_1, ..., i_k, j_1, ..., j_k \in S$ and $i_x \perp j_x$ $(x = 1, 2...k)$ and $f(0) = 0$. Then there exists a unique Orthogonally Additive mapping $\mathcal{A} : S \to T$ such that

$$\sup_{k\in\mathbb{N}} \|(\zeta(i_1) - \mathcal{A}(i_1), ..., \zeta(i_k) - \mathcal{A}(i_k))\|_k \leq \eta \tag{5}$$

$i_1, i_2, ..., i_k \in S$.

Proof. Let $\Lambda = \{g : S \to T | g(0) = 0\}$ and introduce the generalized metric d defined on Λ by

$$d(u, v) = \inf \left\{ \lambda \in [0, \infty] | \sup_{k\in\mathbb{N}} \|(u(j_1) - v(j_1), ..., u(j_k) - v(j_k))\|_k \leq \lambda \quad \forall \quad j_1, ..., j_k \in S \right\}$$

Then it is easy to show that (Λ, d) is a generalized complete metric space [11].
We define an operator $\mathcal{J} : \Lambda \to \Lambda$ by

$$\mathcal{J}u(j) = \frac{1}{2}u(2j) \qquad j \in S$$

We assert that $\mathcal{J}$ is a strictly contractive operator. Given $u, v \in \Lambda$, let $\lambda \in [0, \infty]$ be an arbitary constant with $d(u, v) \leq \lambda$. By the definition

$$\sup_{k\in\mathbb{N}} \|(u(j_1) - v(j_1), ..., u(j_k) - v(j_k))\|_k \leq \lambda \quad j_1, ..., j_k \in S.$$

Therefore,

$$\begin{aligned}
&\sup_{k\in\mathbb{N}} \|(\mathcal{J}u(j_1)-\mathcal{J}v(j_1),...,\mathcal{J}u(j_k)-\mathcal{J}v(j_k))\|_k \\
&\quad\le \sup_{k\in\mathbb{N}} \left\|\left(\frac{1}{2}u(2j_1)-\frac{1}{2}v(2j_1),...,\frac{1}{2}u(2j_k)-\frac{1}{2}v(2j_k)\right)\right\|_k \\
&\quad\le \frac{1}{2}\lambda
\end{aligned}$$

$j_1,...,j_k \in S$. Hence,it holds that

$$d(\mathcal{J}u,\mathcal{J}v) \le \frac{1}{2}\lambda d(\mathcal{J}u,\mathcal{J}v) \le \frac{1}{2}d(u,v)$$

$\forall u, v \in \Lambda$.

Letting $j_1 = j_2 =, ..., = j_k = 0$ in (4), we obtain that

$$\sup_{k\in\mathbb{N}} \|(\zeta(2j_1)-2\zeta(j_1),...,\zeta(2j_k)-2\zeta(2j_k))\|_k \le \eta \tag{6}$$

for all $i_x \in S, i_x \perp 0 \quad (x = 1, 2, ..., k)$.

Dividing on both sides 2 by (6), we can get

$$\sup_{k\in\mathbb{N}} \left\|\left(\zeta(j_1)-\frac{1}{2}\zeta(2j_1),...,\zeta(j_k)-\frac{1}{2}\zeta(2j_k)\right)\right\|_k \le \frac{1}{2}\eta \tag{7}$$

This Means that $\mathcal{J}$ is strictly contractive operator on Λ with the Lipschitz constant $\mathcal{L} = \frac{1}{2}$.

By (7), we have $d(\mathcal{J}\zeta, \zeta) \le \frac{1}{2}\eta < \infty$. According to Theorem 1.1, we deduce the existence of a fixed point of $\mathcal{J}$ that is the existence of mapping $\mathcal{A} : S \to T$ such that

$$\mathcal{A}(2j) = 2\mathcal{A}(j) \qquad \forall j \in S.$$

Moreover, we have $d(\mathcal{J}^n\zeta, \mathcal{A}) \to 0$, which implies

$$\mathcal{A}(q) = \lim_{n\to\infty} \mathcal{J}^n\zeta(j) = \lim_{n\to\infty} \frac{\zeta(2^n j)}{2^n}$$

for all $q \in S$.

Also, $d(\zeta, \mathcal{A}) \le \frac{1}{1-\mathcal{L}} d(\mathcal{J}\zeta, \zeta)$ implies the inequality

$$
\begin{aligned}
d(\zeta, \mathcal{A}) &\leq \frac{1}{1-\frac{1}{2}} d(\mathcal{J}\zeta, \zeta) \\
&\leq \eta.
\end{aligned}
$$

Considering Definition 1.5, we have $2^n i \perp 2^n j$. Set

$$i_1 =, ..., = i_k = 2^n i, j_1 =, ..., = j_k = 2^n j$$

in (4) and divide both sides by 2^n. Then, using property (a) of multi-norms, we obtain

$$
\begin{aligned}
\|D\mathcal{A}(i,j)\| &= \lim_{n\to\infty} \frac{1}{2^n} \|D\zeta\left(2^n i, 2^n j\right)\| \\
&\leq \lim_{n\to\infty} \frac{\eta}{2^n} = 0
\end{aligned}
$$

for all $i, j \in S$. Hence $\mathcal{A}$ is Additive.

The uniqueness of $\mathcal{A}$ follows from the fact that $\mathcal{A}$ is the unique fixed point of $\mathcal{J}$ with the property that there exists $\ell \in (0, \infty)$ such that

$$\sup_{k\in\mathbb{N}} \|(\zeta(i_1) - \mathcal{A}(i_1), ..., \zeta(i_k) - \mathcal{A}(i_k))\|_k \leq \ell$$

for all $i_1, ..., i_k \in S$.

This completes the proof of the Theorem. □

Theorem 1.8. *Let S be an orthogonality space and let $\left((T^k, \|.\|) : K \in \mathbb{N}\right)$ be a multi-Banach space. Suppose that η is a nonnegative real number and $\zeta : S \to T$ is a mapping satisfying the inequality (4). Then there exists a unique Orthogonally Quadratic mapping $\mathcal{Q} : S \to T$ such that*

$$\sup_{k\in\mathbb{N}} \|(\zeta(i_1) - \mathcal{Q}(i_1), ..., \zeta(i_k) - \mathcal{Q}(i_k))\|_k \leq \frac{1}{3}\eta \tag{8}$$

$i_1, i_2, ..., i_k \in S$.

Proof. By (6), we obtain

$$\|(\zeta(2i_1) - 4\zeta(i_1), ..., \zeta(2i_k) - 4\zeta(i_k))\|_k \leq \eta \tag{9}$$

Dividing on both side 4 by (9), we can get

$$\left\|\left(\zeta(i_1) - \frac{1}{4}\zeta(2i_1), ..., \zeta(i_k) - \frac{1}{4}\zeta(2i_k)\right)\right\|_k \leq \frac{1}{4}\eta \tag{10}$$

By (10), we have we have $d(\mathcal{J}\zeta, \zeta) \leq \frac{1}{4}\eta < \infty$.

Also, $d(\zeta, \mathcal{Q}) \leq \frac{1}{1-\mathcal{L}} d(\mathcal{J}\zeta, \zeta)$ implies the inequality

$$\begin{aligned} d(\zeta, \mathcal{Q}) &\leq \frac{1}{1 - \frac{1}{4}} d(\mathcal{J}\zeta, \zeta) \\ &\leq \frac{1}{3}\eta. \end{aligned}$$

The rest of the proof is similar to that of Theorem 1.7. □

Theorem 1.9. *Let S be an an orthogonality space and let $\big((T^k, \|.\|) : K \in \mathbb{N}\big)$ be a multi-Banach space. Suppose that $\eta \geq 0$ and $\zeta : S \to T$ is an mapping satisfying*

$$\sup_{k\in\mathbb{N}} \|(D\zeta(i_1, j_1), ..., D\zeta(i_k, j_k))\|_k \leq \eta \tag{11}$$

$\forall i_1, ..., i_k, j_1, ..., j_k \in S$. Then there exist a unique orthogonally additive mapping $\mathcal{A} : S \to T$ and a unique orthogonally quadratic mapping $\mathcal{Q} : S \to T$ such that

$$\sup_{k\in\mathbb{N}} \|(\zeta(i_1) - \mathcal{A}(i_1) - \mathcal{Q}(i_1), ..., \zeta(i_k) - \mathcal{A}(i_k) - \mathcal{Q}(i_k))\|_k \leq \frac{4}{3}\eta \tag{12}$$

$\forall i_1, i_2, ..., i_k \in S$.

Proof. By Theorem 1.7, 1.8 there exist a unique additive mapping $\mathcal{A}_0 : S \to T$ and a unique quadratic mapping $\mathcal{Q}_0 : S \to T$ such that

$$\sup_{k\in\mathbb{N}} \|(\zeta(i_1) - \mathcal{A}_0(i_1), ..., \zeta(i_k) - \mathcal{A}_0(i_k))\|_k \leq \eta \tag{13}$$

and

$$\sup_{k\in\mathbb{N}} \|(\zeta(i_1) - \mathcal{Q}_0(i_1), ..., \zeta(i_k) - \mathcal{Q}_0(i_k))\|_k \leq \frac{1}{3}\eta \tag{14}$$

for all $i_1, ..., i_k \in S$. Now from (13) and (14), we get

$$\sup_{k\in\mathbb{N}} \|(\zeta(i_1) + \mathcal{A}_0(i_1) - \mathcal{Q}_0(i_1), ..., \zeta(i_k) + \mathcal{A}_0(i_k) - \mathcal{Q}_0(i_k))\|_k \leq \frac{4}{3}\eta \tag{15}$$

for all $i_1, ..., i_k \in S$. Thus we obtain (12) by defining $\mathcal{A}(i) = -\mathcal{A}_0(i)$ and $\mathcal{Q}(i) = \mathcal{Q}_0(i)$. The uniqueness of $\mathcal{A}$ and $\mathcal{Q}$ is easy to show. □

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