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Higher Order Fibonacci Sequence and Series by Generalized Higher Order Variable Co-Efficient Difference Operator

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Abstract. In this paper, we introduce generalized m^{th} order difference operator with variable co-efficient and its inverse by which we obtain higher Fibonacci sequence and its sum. Some theorems and interesting results on the sum of the terms ofhigher Fibonacci sequence with variable co-efficients are derived. Suitable examples are provided to illustrate our results.

Key words: Generalized difference operator, Variable co-efficients, Fibonacci sequence , Closed form solution, Fibonacci summation.

AMS Subject classification: 39A70, 39A10, 47B39, 65J10, 65Q10.

1. Introduction

In 1984, Jerzy Popenda [5] introduced a particular type of difference operator on $u(k)$ as $\Delta_\alpha u(k) = u(k+1) - \alpha u(k)$. In 1989, Miller and Rose [8] introduced the discrete analogue of the Riemann-Liouville fractional derivative and its inverse $\Delta_h^{-\nu} f(t)$ ([1, 4]).

In 2011, M.Maria Susai Manuel, et.al, [7] extended the operator Δ_α to generalized $\alpha-$difference operator as $\underset{\alpha(\ell)}{\Delta} v(k) = v(k+\ell) - \alpha v(k)$ for the real valued function $v(k)$. In 2014, G.Britto Antony Xavier, et.al, [2] introduced $q-$difference operator as $\Delta_q v(k) = v(qk) - v(k)$, $q \in (0, \infty)$ and obtained finite series solution to the corresponding generalized $q-$difference equation $\Delta_q v(k) = u(k)$. With this backround, in this paper, we obtain advanced Fibonacci sequence and its sum by introducing n^{th}-order difference operator with variable co-efficients.

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2. Higher Order Finonacci Sequence And Series By Generalized m^{th} order variable Co-efficient Difference Equation

Fibonacci and Lucas numbers cover a wide range of interest in modern mathematics as they appear in the comprehensive works of Koshy [6] and Vajda [10]. The $k-$Fibonacci sequence introduced by Falcon and Plaza [3] depends only on one integer parameter k and is defined as

$F_{k,0} = 0, \quad F_{k,1} = 1 \quad \text{and} \quad F_{k,n+1} = kF_{k,n} + F_{k,n-1}, \quad \text{where} \quad n \geq 1,\, k \geq 1.$

In particular, if $k = 2$, the Pell sequence is obtained as

$P_0 = 0, \quad P_1 = 1 \quad \text{and} \quad P_{n+1} = 2P_n + P_{n-1} \quad \text{for} \quad n \geq 1.$

Here we introduce m^{th}-order generalized difference operator with variable co-efficients $\underset{\lambda_\alpha(\ell)}{\Delta} v(k) = v(k) - \sum\limits_{i=1}^{n} \alpha_i k^{r_i} v(k - i\ell)$, $\lambda_\alpha(\ell) = (\alpha_1 k^{r_1}, \alpha_2 k^{r_2} ... \alpha_m k^{r_m})$ which generates higher order Fibonacci sequence and its sum.

Definition 2.1. For $k \in [0, \infty)$, higher order Fibonacci sequence is defined as

$$F_0 = 1,\ \ F_1 = \alpha_1 k^{r_1},\ \ F_n = \alpha_1[k-(n-1)\ell]^{r_1} F_{n-1} + \alpha_2[k-(n-2)\ell]^{r_2} F_{n-2},\ \ n \geq 2 \quad (1)$$

If $\alpha_1 = \alpha_2 = r_1 = r_2 = \ell = 1$ in (1), we have the well known Fibonacci sequence.

Example 2.2. *(i) Taking $k = 7$,$\alpha_1 = 10$, $\alpha_2 = 7$, $r_1 = 3$ and $r_2 = 2$ in (1), we get a Fibonacci sequence* $\Big\{1, 490, 193207, 12173560, \cdots\Big\}$.
(ii) When $k = 9$,$\alpha_1 = 0.8$, $\alpha_2 = 0.3$, $r_1 = 2$ and $r_2 = 4$ in (1), we have a Fibonacci sequence $\Big\{1, 583.2, 238903.02, 65566186.13, \cdots\Big\}$.

Similarly, one can obtain higher order Fibonacci sequences corresponding to each $\lambda_\alpha(\ell) = (\alpha_1 k^{r_1}, \alpha_1 k^{r_2} ... \alpha_m k^{r_m}) \in \mathbb{R}^2$.

Definition 2.3. A generalized m^{th} order difference operator with variable co-efficients on $v(k)$, denoted as $\underset{\lambda_\alpha(\ell)}{\Delta} v(k)$, where $\lambda_{\alpha(\ell)} = (\alpha_1 k^{r_1}, \alpha_1 k^{r_2} ... \alpha_m k^{r_m})$ is defined as

$$\underset{\lambda_\alpha(\ell)}{\Delta} v(k) = v(k) - \sum_{i=1}^{m} \alpha_i k^{r_i} v(k - i\ell), \quad k, \ell \in [0, \infty) \quad (2)$$

and its inverse is defined as below;

$$\text{if} \quad \underset{\lambda_{\alpha(\ell)}}{\Delta} v(k) = u(k), \quad \text{then we write} \quad v(k) = \underset{\lambda_\alpha(\ell)}{\overset{-1}{\Delta}} u(k). \quad (3)$$

Lemma 2.4. *Let $v(k)$ be a functions of $k \in (-\infty, \infty)$. Then we obtain*

$$\underset{\lambda_\alpha(\ell)}{\overset{-1}{\Delta}} a^{sk}\Big[1 - \sum_{i=1}^{m} \frac{\alpha_i k^{r_i}}{a^{is\ell}}\Big] = a^{sk}. \tag{4}$$

Proof. Taking $v(k) = a^{sk}$ in (2), we obtain $\underset{\lambda_\alpha(\ell)}{\Delta} a^{sk} = a^{sk}\Big[1 - \sum_{i=1}^{m} \frac{\alpha_i k^{r_i}}{a^{is\ell}}\Big]$. Now (4) follows from (3). □

Corollary 2.5. *If $m = 3$ in lemma 2.4, then we obtained*

$$\underset{\lambda_\alpha(\ell)}{\overset{-1}{\Delta}} a^{sk}\Big[1 - \sum_{i=1}^{3} \frac{\alpha_i k^{r_i}}{a^{is\ell}}\Big] = a^{sk}. \tag{5}$$

Proof. Taking $u(k) = a^{sk}\Big[1 - \sum_{i=1}^{3} \frac{\alpha_i k^{r_i}}{a^{is\ell}}\Big]$ in (2), we have

$$\underset{\lambda_\alpha(\ell)}{\Delta} a^{sk} = a^{sk}\Big[1 - \sum_{i=1}^{3} \frac{\alpha_i k^{r_i}}{a^{is\ell}}\Big].$$

Now (5) follows from (3). □

Corollary 2.6. *Let e^{-sk} be a function of $k \in (-\infty, \infty)$. Then we have*

$$\underset{\lambda_\alpha(\ell)}{\overset{-1}{\Delta}} e^{-sk}\Big[1 - \sum_{i=1}^{m} \alpha_i k^{r_i} e^{is\ell}\Big] = e^{-sk}. \tag{6}$$

Proof. The proof follows by assuming $a = e^{-1}$ in (4). □

Corollary 2.7. *Let e^{-sk} be a function of $k \in (-\infty, \infty)$. Then we have*

$$\underset{\lambda_\alpha(\ell)}{\overset{-1}{\Delta}} e^{-sk}\Big[1 - \sum_{i=1}^{3} \alpha_i k^{r_i} e^{is\ell}\Big] = e^{-sk}. \tag{7}$$

Proof. The proof follows by assuming $m = 3$ in corollary 2.6. □

Corollary 2.8. *Let e^{sk} be a function of $k \in (-\infty, \infty)$, then we obtained*

$$\underset{\lambda_\alpha(\ell)}{\overset{-1}{\Delta}} e^{sk}\Big[1 - \sum_{i=1}^{m} \frac{\alpha_i k_i^{r}}{e^{is\ell}}\Big] = e^{sk}. \tag{8}$$

Proof. The proof follows by taking $a = e^{sk}$ in lemma 2.4. □

Corollary 2.9. *Let e^{sk} be a function of $k \in (-\infty, \infty)$, then we obtained*

$$\overset{-1}{\underset{\lambda_\alpha(\ell)}{\Delta}} e^{sk}\Big[1 - \sum_{i=1}^{3} \frac{\alpha_i k_i^r}{e^{is\ell}}\Big] = e^{sk}. \tag{9}$$

Proof. The proof follows by taking $m = 3$ in corollary 2.8. □

Corollary 2.10. *Let $logk$ be a function of $k > 2\ell$. Then we have*

$$\overset{-1}{\underset{\lambda_\alpha(\ell)}{\Delta}} \Big[logk - \sum_{i=1}^{m} \alpha_i k^{r_i} log(k - i\ell)\Big] = logk. \tag{10}$$

Proof. Taking $v(k) = logk$ in (2), we obtained

$$\underset{\lambda_\alpha(\ell)}{\Delta} logk = logk - \sum_{i=1}^{m} \alpha_i k_i^r log(k - i\ell).$$

Now (10) follows from (3). □

Corollary 2.11. *Let $logk$ be a function of $k > 2\ell$. Then we have*

$$\overset{-1}{\underset{\lambda_\alpha(\ell)}{\Delta}} \Big[logk - \sum_{i=1}^{3} \alpha_i k^{r_i} log(k - i\ell)\Big] = logk. \tag{11}$$

Proof. The proof follows by taking $m = 3$ in corollary 2.10. □

Theorem 2.12. *If $v(k) = \overset{-1}{\underset{\lambda_\alpha(\ell)}{\Delta}} u(k)$, $F_0 = 1, F_1 = F_0\alpha_1 k^{r_1}$ and $F_{n+1} = \sum_{i=0}^{n} F_{n-i}\alpha_{i+1}[k-(n-i)\ell]^{r_{i+1}}$, then* $\sum_{i=0}^{n} F_i\, u(k-i\ell) = v(k) - F_{n+1}v(k-(n+1)\ell) -$

$$\sum_{i=0}^{n} F_{n-i}\alpha_{i+2}[k-(n-i)\ell]^{r_{i+2}} v(k-(n+2)\ell) -$$

$$\sum_{i=0}^{n-1} F_{n-i}\alpha_{i+3}[k-(n-i)\ell]^{r_{i+3}} v(k-(n+3)\ell) + ... + F_n\alpha_m(k-n\ell)^{r_m} v(k-(n+m)\ell). \tag{12}$$

Proof. From (2) and (3), we arrive

$$v(k) = u(k) + \alpha_1 k^{r_1} v(k-\ell) + \alpha_2 k^{r_2} v(k-2\ell) + .. + \alpha_m k^{r_m} v(k-m\ell). \tag{13}$$

Replacing k by $k-\ell$ and then substituting the value of $v(k-\ell)$ in (13), we get

$$v(k) = u(k) + F_1 u(k-\ell) + [F_1\alpha_1(k-\ell)^{r_1} + \alpha_2 k^{r_2}]v(k-2\ell) + ...+$$

$$[F_1\alpha_{m-1}(k-\ell)^{r_{m-1}} + \alpha_m k^{r_m}]v(k-m\ell) + F_1\alpha_m(k-\ell)^{r_m} v(k-(m+1)\ell) \tag{14}$$

which gives

$$v(k) = u(k) + F_1u(k-\ell) + F_2v(k-2\ell) + [F_1\alpha_2(k-\ell)^{r_2} + \alpha_3k^{r_3}]v(k-3\ell) + ...+$$

$$[F_1\alpha_{m-1}(k-\ell)^{r_{m-1}} + \alpha_mk^{r_m}]v(k-m\ell) + F_1\alpha_m(k-\ell)^{r_m}v(k-(m+1)\ell), \quad (15)$$

where F_0, F_1 and F_2 are given in (1).

Replacing k by $k-2\ell$ in (13) and then substituting $v(k-2\ell)$ in (15), we obtain

$$v(k) = \sum_{i=0}^{3} F_iu(k-i\ell) + ... + F_2\alpha_m(k-2\ell)^{r_m}v(k-(m+2)\ell),$$

where F_3 is given in (1). Repeating this process again and again, we get (16). □

Corollary 2.13. *If* $v(k) = \overset{-1}{\underset{\lambda_\alpha(\ell)}{\Delta}} u(k)$, $F_0 = 1, F_1 = F_0\alpha_1k^{r_1}$ *and* $F_{n+1} = \sum_{i=0}^{n} F_{n-i}\alpha_{i+1}[k-(n-i)\ell]^{r_{i+1}}$, *then* $\sum_{i=0}^{n} F_i\, u(k-i\ell) = v(k)-F_{n+1}v(k-(n+1)\ell)-$

$$\sum_{i=0}^{1} F_{n-i}\alpha_{i+2}[k-(n-i)\ell]^{r_{i+2}}v(k-(n+2)\ell) + F_n\alpha_3(k-n\ell)^{r_3}v(k-(n+3)\ell). \quad (16)$$

Proof. The proof follows by taking $m = 3$ in Theorem 2.12. □

Corollary 2.14. *If $v(k)$ is a closed form solution of the m^{th} order generalized difference equation*

$$\underset{\lambda_\alpha(\ell)}{\Delta} v(k) = a^{sk}\Big[1 - \frac{\alpha_1k^{r_1}}{a^{s\ell}} - \frac{\alpha_2k^{r_2}}{a^{2s\ell}} - \frac{\alpha_1k^{r_3}}{a^{3s\ell}}\Big],$$

then we obtain

$$a^{sk}\Big[1 - \frac{F_{n+1}}{a^{s(n+1)\ell}} - \sum_{i=0}^{1}\frac{F_{n-i}\alpha_{i+2}[k-(n-i)\ell]^{r_{i+2}}}{a^{s(n+2)\ell}} - \frac{F_n\alpha_3(k-n\ell)^{r_3}}{a^{s(n+3)\ell}}\Big]$$

$$= \sum_{i=0}^{n} F_ia^{s(k-i\ell)}\Big[1 - \frac{\alpha_1(k-i\ell)^{r_1}}{a^{s\ell}} - \frac{\alpha_2(k-i\ell)^{r_2}}{a^{2s\ell}} - \frac{\alpha_3(k-i\ell)^{r_3}}{a^{3s\ell}}\Big]. \quad (17)$$

Proof. The proof of (17) follows by taking $v(k) = a^{sk}$ and applying (4) in (16). □

The following example is an verification of corollary 2.14.

Example 2.15. *Taking* $k = 9$,$\ell = 0.3$,$n = 1$, $a = 5$, $\alpha_1 = 0.2$, $\alpha_2 = 0.3$,$\alpha_3 = 0.4$, $r_1 = 1$ *and* $r_2 = 3$,$r_3 = 4$ *in* (17)*, we get*

$$5^9 - F_25^{-2} - 3F_25^{-5} = \sum_{i=0}^{1} F_i5^{(9-0.3i)}\Big[1 - \tfrac{2(7-3i)^1}{5^3} - \tfrac{3(7-3i)^2}{5^6}\Big] = 78077.15136,$$

where $F_0 = 1$, $F_1 = 14$, $F_2 = 259$, $F_3 = 1190$.

Corollary 2.16. *Let e^{-sk} be a function of $k \in (-\infty, \infty)$. Then*

$$e^{-sk}\Big[1 - F_{n+1}e^{s(n+1)\ell} - \sum_{i=0}^{1} F_{n-i}\alpha_{i+2}[k-(n-i)\ell]^{r_{i+2}}e^{s(n+2)\ell} + F_n\alpha_3(k-n\ell)^{r_3}e^{s(n+3)\ell}\Big]$$

$$= \sum_{i=0}^{n} F_i e^{-s(k-i\ell)}\Big[1 - \alpha_1(k-i\ell)^{r_1}e^{s\ell} - \alpha_2(k-i\ell)^{r_2}e^{2s\ell} - \alpha_3(k-i\ell)^{r_3}e^{3s\ell}\Big]. \quad (18)$$

Proof. Taking $v(k) = e^{-sk}$ and applying (6) in (4), we get (18). □

Example 2.17. *Taking $k = 9$, $\ell = 1$, $n = 3$, $\alpha_1 = 0.8$, $\alpha_2 = 0.3$, $r = 3$ and $s = 2$ in* (18)*, then we obtained*

$$e^{-9} - F_4e^5 - (0.3)6^2F_3e^{-4} = \sum_{i=0}^{3} F_i e^{-(9-i)}\Big[1 - (0.8)(9-i)^3e - (0.3)(9-i)^2e^2\Big] = -89333078.94$$

where $F_0 = 1$, $F_1 = 583.2$, $F_2 = 238903.02$, $F_3 = 65566186.13$ and $F_4 = 11333348840$.

Theorem 2.18. *Let $t \in \mathbb{N}(0)$. Then a closed form solution of the generalized m^{th} order difference equation $v(k) - \sum_{i=1}^{m} \alpha_i k^{r_i} v(k-i\ell) = \Big[k^t - \sum_{i=1}^{m} \alpha_i k^{r_i}(k-i\ell)^t\Big]$ is*

$$\mathop{\Delta}_{\lambda_\alpha(\ell)}^{-1}\Big[k^t - \sum_{i=1}^{m} \alpha_i k^{r_i}(k-i\ell)^t\Big] = k^t \quad (19)$$

Proof. Taking $v(k) = k^t$ in (2) and using (3), we get (19). □

Corollary 2.19. *If $v(k) = \mathop{\Delta}_{\lambda_\alpha(\ell)}^{-1}\Big[k^t - \sum_{p=1}^{m} \alpha_p k^{r_p}(k-p\ell)^t\Big]$ is the closed form solution given in (19), then*

$$v(k) - F_{n+1}(k-(n+1)\ell)^t - \sum_{i=0}^{n} F_{n-i}\alpha_{i+2}[k-(n-i)\ell]^{r_{i+2}}(k-(n+2)\ell)^t + ...+$$

$$F_n\alpha_m(k-n\ell)^{r_m}(k-(n+m)\ell)^t = \sum_{i=0}^{n} F_i\Big[(k-i\ell)^t - \sum_{p=1}^{m}\alpha_p(k-p\ell)^{r_p}[k-(i+p)\ell]^t\Big]. \quad (20)$$

Proof. Taking $u(k) = k^t - \sum_{p=1}^{m} \alpha_p k^{r_p}(k-p\ell)^t$ in Theorem 2.12, we have 20. □

Example 2.20. *Let $k = 7$, $\ell = 2$, $n = 3$, $t = 2$, $r = 3$, $s = 4$ $\alpha_1 = 5$, $\alpha_2 = 3$ in Corollary* (2.19)*. Then*

$$\sum_{i=0}^{3} F_i u(7-2i) = v(7) - F_4v(-1) - \alpha_2F_3v(-3) = -5,026,731,585.$$

where $u(k) = k^t - \alpha_1k^r(k-\ell)^t - \alpha_2k^s(k-2\ell)^t$, $F_0 = 1$, $F_1 = 1715$, $F_2 = 1,079,078$, $F_3 = 148,891,115$ and $F_4 = 1,006,671,529$.

Corollary 2.21. *If $v(k)$ is a closed form solution of m^{th} order difference equation with variable co-efficients*
$v(k) - \sum_{i=1}^{m} \alpha_i k^{r_i} v(k - i\ell) = k^t a^{sk} - \sum_{i=1}^{m} \Big[\alpha_i k^{r_i}(k - i\ell)^t a^{s(k-i\ell)}\Big]$, *then we have*

$$k^t a^{sk} - F_{n+1}(k - (n+1)\ell)^t a^{s(k-(n+1)\ell)} - \sum_{i=0}^{n} F_{n-i}\alpha_{i+2}[k - (n-i)\ell]^{r_{i+2}} \times$$
$$(k - (n+2)\ell)^t a^{s(k-(n+2)\ell)} + ... + F_n \alpha_m (k - n\ell)^{r_m} (k - (n+m)\ell)^t a^{s(k-(n+m)\ell)}$$

$$= \sum_{i=0}^{n} F_i \Big[(k - i\ell)^t - \sum_{p=1}^{m} \alpha_p (k - p\ell)^{r_p} [k - (i+p)\ell]^t a^{s[k-(i+p)\ell]}\Big]. \tag{21}$$

Proof. Taking $u(k) = k^t a^{sk} - \sum_{i=1}^{m} \Big[\alpha_i k^{r_i}(k - i\ell)^t a^{s(k-i\ell)}\Big]$ in Theorem 2.12 and using (4), we get 21. □

Corollary 2.22. *A closed form solution of generalized third odrer difference equation* $\underset{\lambda_\alpha(\ell)}{\Delta} v(k) = k^2 a^{sk} - \sum_{i=1}^{3} \Big[\alpha_i k^{r_i}(k - i\ell)^2 a^{s(k-i\ell)}\Big]$ *is $k^2 a^k$ and hence we have*

$$k^2 a^{sk} - F_{n+1}(k - (n+1)\ell)^2 a^{s(k-(n+1)\ell)} - \sum_{i=0}^{1} F_{n-i}\alpha_{i+2}[k - (n-i)\ell]^{r_{i+2}}$$
$$(k - (n+2)\ell)^2 a^{s(k-(n+2)\ell)} + F_n \alpha_3 (k - n\ell)^{r_3} (k - (n+3)\ell)^2 a^{s(k-(n+3)\ell)}$$

$$= \sum_{i=0}^{n} F_i \Big[(k - i\ell)^2 a^{s(k-i\ell)} - \sum_{p=1}^{3} \alpha_p (k - i\ell)^{r_p} [k - (p+i)\ell]^2 a^{s(k-(p+i)\ell)}\Big]. \tag{22}$$

Proof. The proof follows by taking $m = 3$ and $t = 2$ in Corollary 2.21. □

Example 2.23. *Let $k = 5$, $\ell = 2$, $a = 3$, $n = 4$, $\alpha_1 = 0.02$, $\alpha_2 = 0.03$, $r = 3$, $s = 2$ in Corollary* (2.22). *Then we obtain*
$v(5) - F_5 v(-5) - (0.03)F_4 v(-7) = \sum_{i=0}^{3} F_i[(5 - 2i)^3 3^{k-2i} - (0.02)(5 - 2i)^3 \times$
$[5 - (i+1)2]^3 3^{5-(i+1)2} - (.03)(5 - 2i)^2 [5 - 2(i+2)]^3 3^{5-2(i+2)}] = 24,611,856.47,$
where $F_0 = 1,\ F_1 = 2.5,\ F_2 = 2.1,\ F_3 = 0.717,\ F_4 = 0.04866$ *and* $F_5 = 0.0477864.$

Corollary 2.24. *A closed form solution of the second order difference equation* $v(k) - \sum_{i=0}^{3} \alpha_i k^{r_i} v(k - i\ell) = k^t e^{-sk} - \sum_{p=0}^{3} \alpha_p k^{r_p}(k - p\ell)^t e^{-s(k-p\ell)}$ *is given by*

$$k^t e^{-sk} - F_{n+1}(k - (n+1)\ell)^t e^{-s(k-(n+1)\ell)} - \sum_{i=0}^{1} F_{n-i}\alpha_{i+2}[k - (n-i)\ell]^{r_{i+2}} \times$$

$$(k-(n+2)\ell)^t e^{-s(k-(n+2)\ell)} + F_n\alpha_3(k-n\ell)^{r_3}(k-(n+3)\ell)^t e^{-s(k-(n+3)\ell)}$$
$$= \sum_{i=0}^{n} F_i e^{-s(k-i\ell)}\Big[(k-i\ell)^t - \sum_{p=1}^{3}\alpha_p(k-(p+i)\ell)^{r_p}[k-(p+i)\ell]^t e^{sp\ell}\Big]. \tag{23}$$

Proof. Taking $a = e^{-1}$ in (21), we get (23). □

Corollary 2.25. *If* $v(k) = \underset{\lambda_\alpha(\ell)}{\overset{-1}{\Delta}}\Big[ke^{-sk} - \sum\limits_{p=1}^{m}\alpha_p k^p(k-p\ell)e^{-s(k-\ell)}\Big]$ *is the closed form solution given in (23), then*

$$ke^{-sk} - F_{n+1}(k-(n+1)\ell)e^{-s(k-(n+1)\ell)} - \sum_{i=0}^{1} F_{n-i}\alpha_{i+2}[k-(n-i)\ell]^{r_{i+2}}\times$$
$$(k-(n+2)\ell)e^{-s(k-(n+2)\ell)} + F_n\alpha_3(k-n\ell)^{r_3}(k-(n+3)\ell)e^{-s(k-(n+3)\ell)}$$
$$= \sum_{i=0}^{n} F_i e^{-s(k-i\ell)}\Big[(k-i\ell) - \sum_{p=1}^{3}\alpha_p(k-(p+i)\ell)^{r_p}[k-(p+i)\ell]e^{sp\ell}\Big]. \tag{24}$$

Proof. The proof follows by taking $t = 1$ in Corollory 2.24. □

Theorem 2.26. *Let $v(k)$ be a solution of the n^{th}-order difference equation with variable co-efficients*

$$v(k) - \sum_{i=0}^{m}\alpha_i k^{r_i} v(k-i\ell) = k^{(t)}a^{sk} - \sum_{p=1}^{m}\alpha_p k^{r_p}(k-p\ell)^{(t)}a^{s(k-p\ell)},$$

then we have

$$k^{(t)}a^{sk} - F_{n+1}(k-[n+1]\ell)^{(t)}a^{s(k-[n+1]\ell)} - \sum_{i=0}^{n} F_{n-i}\alpha_{i+2}[k-(n-i)\ell]^{r_{i+2}}\times$$
$$(k-[n+2]\ell)^{(t)}a^{s(k-[n+2]\ell)} - \sum_{i=0}^{n-1} F_{n-i}\alpha_{i+3}[k-(n-i)\ell]^{r_{i+3}}(k-[n+3]\ell)^{(t)}a^{s(k-[n+3]\ell)}$$
$$+ ... + F_n\alpha_m(k-n\ell)^{r_m}(k-[n+m]\ell)^{(t)}a^{s(k-[n+m]\ell)}$$
$$= \sum_{i=0}^{n} F_i a^{s(k-i\ell)}\Big[(k-i\ell)^{(t)} - \sum_{p=1}^{m}\alpha_p(k-i\ell)^{r_p}(k-(i+p)\ell)^{(t)}a^{-sp\ell)}\Big] \tag{25}$$

Proof. Taking $v(k) = k^{(t)}a^{sk}$ in Theorem 2.12 and using (4), we get 25. □

Corollary 2.27. *If $v(k)$ is the closed form solution given of (25), then*

$$k^{(2)}a^{sk} - F_{n+1}(k-[n+1]\ell)^{(2)}a^{s(k-[n+1]\ell)} - \sum_{i=0}^{1} F_{n-i}\alpha_{i+2}[k-(n-i)\ell]^{r_{i+2}}\times$$
$$(k-(n+2)\ell)^{(2)}a^{-s(k-(n+2)\ell)} - F_n\alpha_3(k-n\ell)^{r_3}(k-(n+3)\ell)e^{-s(k-(n+3)\ell)}$$
$$= \sum_{i=0}^{n} F_i a^{s(k-i\ell)}\Big[(k-i\ell)^{(2)} - \sum_{p=1}^{3}\alpha_p(k-i\ell)^{r_p}[k-(i+p)\ell]^{(2)}a^{-ps\ell}\Big]. \tag{26}$$

Proof. The proof follows by taking $m = 3$ and $t = 2$ in Theorem 2.26. $\square$

Example 2.28. *Let $k = 7$, $\ell = 2$, $a = 3$, $n = 2$, $\alpha_1 = 0.04$, $\alpha_2 = 0.06$, $r = 4$, $s = 3$ in Corollary (2.27). Then we obtain*

$$v(7) - F_3 v(1) - (0.06)3^3 F_2 v(-1) = \sum_{i=0}^{2} F_i[(7-2i)^{(2)}3^{k-2i} - (0.06)(7-2i)^3$$
$$[7-2(i+1)]^{(2)}3^{7-2(i+1)} - (.06)(7-2i)^{(2)}[7-2(i+2)]^3 3^{7-2(i+2)}] = 84008.0808,$$

where $F_0 = 1$, $F_1 = 96.04$, $F_2 = 2421.58$, $F_3 = 8566.2192$.

Corollary 2.29. *Let $v(k)$ be a solution of n^{th} order difference equation with variable co-efficients $v(k) - \sum_{i=0}^{m} \alpha_i k^{r_i} v(k - i\ell) = e^{-sk}\Big[k^{(2)} - \sum_{p=1}^{m} \alpha_p k^{r_p}(k - p\ell)^{(2)} e^{ps\ell}\Big]$. Then we have*

$$k^{(2)}e^{-sk} - F_{n+1}(k - [n+1]\ell)^{(2)}e^{-s(k-[n+1]\ell)} - \sum_{i=0}^{1} F_{n-i}\alpha_{i+2}[k-(n-i)\ell]^{r_{i+2}} \times$$
$$(k-(n+2)\ell)^{(2)}e^{-s(k-(n+2)\ell)} - F_n\alpha_3(k-n\ell)^{r_3}(k-(n+3)\ell)^{(2)}e^{-s(k-(n+3)\ell)}$$
$$= \sum_{i=0}^{n} F_i e^{-(k-i\ell)}\Big[(k-i\ell)^{(2)} - \sum_{p=1}^{m}\alpha_p(k-i\ell)^{r_p}[k-(i+p)\ell]^{(2)}e^{ps\ell}\Big]. \quad (27)$$

Proof. Taking $a = e^{-1}$ in (2.27), we get (27). $\square$

Example 2.30. *Let $k = 6$,$\ell = 0.21$ $n = 2$, $a = 0.2$, $\alpha_1 = 2$, $\alpha_2 = 0.3$, $r = 3$, $s = 2$ in Corollary* (2.29). *Then we obtain*

$$v(6) - F_3 v(5.37) - (0.3)(5.58)^2 F_2 v(5.16) = \sum_{i=0}^{3} F_i[(6-(0.21)i)^{(3)}(0.2)^{k-(0.21)i} -$$
$$(2)(6-(0.21)i)^3[6-(0.21)(i+1)]^{(3)}(0.2)^{6-(0.21)(i+1)} - (.3)(6-(0.21)i)^2$$
$$[6-(0.21)(i+2)]^{(3)}3^{6-(0.21)(i+2)}] = -7,539,276.7060093,$$

where $F_0 = 1$, $F_1 = 432$, $F_2 = 167717.1217$ and $F_3 = 8746152.49$.

Conclusion: We obtained summation formula to Higher order Fibonacci sequence by introducing generalized m^{th} order difference operator with variable co-efficients and have derived certain results on closed and summation form solution of generalized m^{th} order difference equation with variable co-efficients which will be used to our further research.

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